Dynamic Adverse Selection: A Theory of Illiquidity, Fire Sales, and Flight to Quality*

Veronica Guerrieri Robert Shimer

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Abstract

We develop a dynamic equilibrium model of asset markets with adverse selection. There exists a unique equilibrium where better quality assets trade at higher prices but with a lower price-dividend ratio in less liquid markets. Sellers of high-quality assets signal their quality by accepting a lower trading probability. We show how the distribution of sellers' private information affects an asset's price and liquidity, how a change in that distribution can cause a fire sale and flight-to-quality, and how asset purchase and subsidy programs may raise prices and liquidity and reverse the flight-to-quality.

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1 Introduction

This paper develops a dynamic equilibrium model of asset markets with adverse selection. The owners of heterogeneous assets are privately informed about the quality of their assets. Sellers set prices for their assets recognizing that sales may be rationed at high prices. Buyers set prices recognizing that the quality of available assets may depend on the price selected. In equilibrium, sellers of high quality assets are willing to set a high price despite the low sale probability because the continuation value from failing to sell a high quality asset is high; conversely, sellers of low quality assets opt for a low price. Buyers are indifferent between paying a low price for a low quality asset and a high price for a high quality asset.

We prove these results in a deliberately stylized dynamic general equilibrium framework. Assets are perfectly durable and pay a constant dividend each period, some amount of a perfectly perishable consumption good. Better quality assets pay a higher dividend but only the asset's current owner observes the dividend. This is the source of private information and the root of the adverse selection problem, as in Akerlof (1970). The only permissible trades are between the consumption good and the asset. Individuals are risk-neutral and have a discount factor that changes over time, independently across individuals, creating gains from trade. Finally, discount factors are observable, which ensures that patient individuals never sell assets since there are no gains from trade. We believe this framework is useful for capturing our main idea that illiquidity may separate high and low quality assets in markets with private information. The dynamic aspects of the model are important because buyers' willingness to pay for an asset depends on the possibility of resale, which creates a liquidity premium in asset prices.

We define two equilibrium concepts in this framework and prove that both are unique. The first is a partial equilibrium, which takes as given the value of a unit of the consumption good to the buyer and ignores the market clearing condition that demand and supply of the consumption good are equal. The second is a competitive equilibrium in which the value of the consumption good is endogenous and the market for the consumption good clears. Key to both equilibrium concepts is that buyers' beliefs about the quality of asset purchased at a particular price must respect sellers' incentive to sell at that price. More precisely, if buyers anticipate getting a particular quality asset with positive probability at a given price, it must be weakly optimal for a seller to offer that quality asset at that price.

Although our model is abstract, we believe it may be useful for understanding and ultimately quantifying the importance of adverse selection for market liquidity. To be concrete, consider the market for AAA-rated private-label mortgage-backed securities (MBS) from 2005 to 2008. At the start of this period, market participants viewed these securities as a safe investment, nearly indistinguishable from a Treasury bond. By 2007, investors started to recognize that some of these securities were likely to pay less than face value. Moreover, it was difficult to determine the exact assets that backed each individual security. Anticipating that she might later have to sell it, the owner of an asset had an incentive to learn its quality. On the other hand, it may not have been profitable for potential buyers to investigate the quality of all possible assets because they did not know which assets would later be for sale. Although we do not model the process of learning about an asset's quality, we view this world with private information and adverse selection as the starting point for our model.

Our model predicts that a seller should always be able to sell an asset at a sufficiently low price. However, the owner of a high quality private-label MBS will choose to hold out for a higher price, despite the shortage of buyers at that price. Moreover, the price that buyers are willing to pay for a high quality security is depressed because the market is less liquid. That is, even if a buyer somehow understood that a particular asset would pay the promised dividends with certainty, he would pay less for it because he would anticipate having trouble reselling it to future buyers who don't have his information. Illiquidity therefore further depresses asset prices. For this reason, we view an event where sellers start to learn the quality of the assets in their portfolio as a fire sale.[1] During a fire sale, buyers still would like to reinvest their income in some asset, and so the decline in the demand for private-label MBS will boost the demand for other assets that do not suffer from an adverse selection problem, such as Treasury bonds. Thus our model generates a flight-to-quality episode, defined as a decrease in the volume of transactions in the security with a fire sale and an increase in the volume and price of alternative investment vehicles.

We demonstrate these ideas formally through a sequence of propositions that derive properties of the partial and competitive equilibrium. First, we consider two *types* of assets. Both buyers and sellers know an asset's type, but only the seller knows its quality. For example, one type of asset may be an agency MBS and another type may be a private-label MBS. Buyers can distinguish between these broad classes of assets, but a seller may have some private information about the particular security that he owns. We first show that a proportionate difference in the assets' payoffs is associated with a proportionate difference in their price and no difference in their liquidity (Proposition 4). This suggests

[1]For a detailed description of the the early stages of the financial crisis and an analysis of the source of the adverse selection problem, see Gorton (2008). This view of the crisis is consistent with Dang, Gorton and Holmström (2009), who conclude, "Systemic crises concern debt. The crisis that can occur with debt is due to the fact that the debt is not riskless. A bad enough shock can cause information insensitive debt to become information sensitive, make the production of private information profitable, and trigger adverse selection. Instead of trading at the new and lower expected value of the debt given the shock, agents trade much less than they could or even not at all. There is a collapse in trade. The onset of adverse selection is the crisis."

that liquidity is instead related to the second moment of asset's quality, i.e. to the extent of the private information problem. A natural conjecture is that if the private information problem is more severe for private-label MBS than agency MBS, in the sense of sense of second order stochastic dominance, then the average price, liquidity (fraction of the asset sold each period), and volume (value of sales in units of the consumption good) in the private-label will be lower. We show that while this is true in some special cases (Proposition 5), it is not generally true (Proposition 6). In particular, if the dividends from the two assets have the same support and the same expected value, then private-label MBS will have a higher average price, liquidity, and volume despite owners having superior information. This leads to one of our key insights: the extent of the adverse selection problem depends critically on the support of the dividend distribution. A reduction in the lower bound of the support has a profound impact on prices, liquidity, and volume, even if the distribution of asset quality is otherwise nearly unchanged.

Second, we turn to fire sales and flight-to-quality. Rather than comparing two types of assets with different quality distributions, we ask what happens if there is a change in the quality of one type of asset, holding fixed the quality distribution of other assets. In partial equilibrium, this question is virtually the same as the comparison in the previous paragraph across assets of different quality, but a change in the distribution of an asset's quality will also have general equilibrium effects. Consider any change in the quality distribution of one type of asset that in partial equilibrium reduces the volume of trade in that asset; we prove that in a competitive equilibrium, it must raise the price, liquidity, and volume of all other assets (Proposition 7). Putting the two sets of results together, we obtain a picture of a fire sale causing a flight-to-quality. For example, a worsening of the private information problem in the market for private-label MBS, such as a reduction in the lower bound of the support of the dividend distribution, causes an increase in the price and volume of other assets, such as agency MBS and U.S. Treasury bonds. Intuitively, investors who used to purchase private-label MBS direct their funds towards other opportunities, raising prices and moderating adverse selection problems in those markets.

Third, we examine how two realistic government interventions affect the market for a particular type of asset. The first is an asset purchase program. The government—or more generally, any player with deep pockets—stands ready to purchase any amount of that type of asset at a specified price. We prove that any asset which, if there were no resale problem would be worth less than that price, is sold to the government, so the government effectively purchases the lowest quality assets. This raises the lower bound on the support of the assets circulating in the private market, which in partial equilibrium raises the price and liquidity of those assets (Propositions 8 and 9). In other words, by ridding the market of the worst assets, an asset purchase program can improve the functioning of the market for those assets that remain in private hands. The general equilibrium impact of the program is ambiguous, since it depends on how the government funds the purchases and what the government does with the dividends from the assets that it purchases. But in principle an asset purchase program can also undo a flight-to-quality episode.

We also consider an asset subsidy program. We now assume that the government subsidizes the purchase of a type of asset at a low price, with the subsidy phasing out as the price increases. We prove that in partial equilibrium, the subsidy program raises the price and liquidity of all assets of that type, including assets that are not subsidized (Propositions 10 and 11). The subsidy program effectively relaxes incentive constraints, making it cheaper for the owner of a high quality asset to signal its quality. This raises the asset's liquidity and hence buyers' valuations. Once again, general equilibrium impacts are ambiguous, since they depend on how the government funds the subsidies.

Finally, we show that the extent of illiquidity does not hinge on assumptions about the frequency of trading opportunities. Even in the limit with continuous trading opportunities, there are not enough buyers in the market for high quality assets and so it takes a real amount of calendar time to sell at a high price. From the perspective of a seller, selling opportunities arrive at a Poisson arrival rate, which sellers wish would be higher. While this may seem similar to the predictions of search theoretic models of illiquidity in asset markets (e.g. Duffie, Gârleanu and Pedersen, 2005; Weill, 2008; Lagos and Rocheteau, 2009), there are important differences. For example, the difficulty of finding a buyer depends primarily on the extent of private information rather than on the availability of trading opportunities. This is because real trading delays are essential for separating the good assets from the bad ones. Of course, in reality adverse selection and search frictions may coexist in a market, and it is indeed straightforward to introduce search into our framework (Guerrieri, Shimer and Wright, 2010; Chang, 2011).

Our model has at least two robust empirical predictions which distinguish it from most of the existing literature on adverse selection. First, we predict that the left tail of the quality distribution is critical for asset prices and liquidity. For example, there is no trade in any type of asset for which the support of the quality distribution includes a zero dividend. This sensitivity recalls the behavior of markets in the presence of Knightian uncertainty, in which traders behave as if they anticipate purchasing the worst possible asset (Rigotti and Shannon, 2005; Caballero and Krishnamurthy, 2008; Routledge and Zin, 2009; Easley and OHara, 2010). The emergence of Knightian uncertainty can similarly cause a collapse in asset prices and trading volumes, although the source of this fragility is very different in our environment. Since in practice the extent of asymmetric information may be small, this observation is important for understanding why asymmetric information matters for equilibrium outcomes.

Second, we predict that sellers are rationed in equilibrium, and would like to sell more at their chosen price. An alternative implementation of our equilibrium would have sellers retain fractional ownership of their assets in order to signal the quality, as in DeMarzo and Duffie (1999). Under this interpretation, a seller who wants to sell a larger fraction of her assets would be forced to accept a lower price, a phenomenon that is usually seen as symptomatic of price pressure in an imperfectly competitive market. We offer an alternative interpretation based on the idea that the price acts as a signal.

Whether adverse selection is important for financial markets is ultimately an empirical question. In practice, it is difficult to measure the extent of adverse selection in any market simply because the data demands are acute. In one of the more successful efforts, Finkelstein and Poterba (2004) find a correlation between characteristics of annuity contracts and characteristics of annuity buyers that are unobserved by annuity sellers. Our model would suggest a similar test in securities markets, a correlation between the frequency that an asset is resold and the asset's terminal payoff conditional on observable characteristics. While our reading of the existing evidence, e.g. Downing, Jaffee and Wallace (2009), suggests that the extent of adverse selection in asset markets is small but positive, it is worth stressing that even a small amount of left tail risk can generate substantial illiquidity in our environment.

Another argument against the relevance of adverse selection in secondary markets is that neither buyers nor sellers knew what they were trading. We are unaware of any direct evidence for (or against) private information in the secondary market, but Arora, Barak, Brunnermeier and Ge (2011) claim that the structure of collateralized debt obligations made it computationally infeasible for anyone but the original issuer to measure the quality of the underlying assets. This is important since in our framework, symmetric lack of information is not a barrier to trade. But our model gives us a reason to believe that adverse selection may still be a problem, despite the computational complexity of unraveling the underlying securities. In equilibrium, prices transmit information from sellers to buyers. Even if the owner of an asset cannot observe an asset's dividend, he knows what he paid for the asset and therefore he knows what value the seller assigned to the asset. Here our model again gives a different perspective from models in which all sales occur at a single price.

A third argument against adverse selection is that markets find solutions to these sorts of problems. One is reputation sustained through repeated interactions between buyers and sellers. In our model, all trade is anonymous so there is no possibility of sustaining a reputation for delivering only high quality assets. We view this as a reasonable description of a crisis episode, even if it is a poor description of the behavior of large financial intermediaries during normal times. When facing solvency constraints, sellers may be willing to sacrifice their long-run reputation for the short-run benefits of liquidating their portfolio. Another market solution is paying a third party to evaluate the quality of assets. Indeed, this is one role that rating agencies are supposed to play. But when a crisis is triggered by the realization that rating agencies had incorrectly assessed risks, there may be no one positioned to offer credible valuation assessments.

A large theoretical literature argues that adverse selection may be important in financial markets. Most papers look at a market structure in which all trades must take place at one price and so generate endogenous illiquidity because sellers may choose not to sell high quality assets if the equilibrium price is too low (e.g. Eisfeldt, 2004; Daley and Green, 2012; Tirole, 2012; Kurlat, Forthcoming; Chari, Shourideh and Zetlin-Jones, 2010). The nature of illiquidity is different in our model: sellers try to sell all their assets at optimally chosen prices, recognizing that sales will be rationed at most prices. This distinction is important for both of our robust empirical predictions. A one price equilibrium is not particularly sensitive to the left tail of the dividend distribution and instead typically requires large changes in the distribution of private information in order to generate large movements in prices and liquidity.[2] And a one price equilibrium cannot explain why a seller would generically choose only to sell fractional ownership in an asset. In any case, these papers typically have a different objective than ours, for example analyzing optimal policy.

A third approach to adverse selection assumes random matching between uninformed buyers and informed sellers and allows the buyers to make take-it-or-leave-it offers to sellers (Inderst, 2005; Chiu and Koeppl, 2011; Camargo and Lester, 2012). Some buyers offer higher prices than others and the owners of high quality assets only sell when they are offered a high price. This implies that the distribution of asset quality across sellers is endogenous, which feeds back into the adverse selection problem. Our approach is different in that it does not depend on an endogenous composition of asset quality. We highlight this by assuming in most of our paper that the fraction of individuals who are sellers and the asset quality distribution owned by those individuals are constant and exogenous because discount factors are independent over time. These papers also do not explore our first robust empirical prediction about the criticality of the left tail of the asset distribution, perhaps because tractability dictates that they focus on environments with only two quality levels. In any case, we would not expect the same tail properties to be relevant in these environments. Nor do these papers find a role for buyers to offer sellers fractional asset purchases, as in DeMarzo and Duffie (1999). Thus they do not uncover our second robust prediction that

[2] The existing literature has not explored this issue in much detail, in part because much of the literature assumes that there are only two quality levels.

sellers are rationed at the equilibrium price. Again, these papers' objective is different, often focusing on optimal policy.

This paper builds on our previous work with Randall Wright (Guerrieri, Shimer and Wright, 2010). It also complements a contemporaneous paper by Chang (2011). There are a number of small differences between that paper and this one. For example, we look at an environment in which individuals may later want to resell assets that they purchase today. This means that buyers care about the liquidity of the asset and so liquidity affects the equilibrium price. It follows that interventions in the market which boost liquidity may also raise asset prices. We also focus explicitly on a general equilibrium environment, allowing for the possibility that buyers may be driven to a corner in which they do not consume anything. This is essential for our model to generate a flight-to-quality. As we discuss in the conclusion, it is also essential to a version model in which individuals' discount factors are unobservable. Still, both papers leverage our earlier research to study separating equilibria in a dynamic adverse selection environment.

This paper proceeds as follows. Section 2 describes our basic model. Section 3 describes the individual's problem and expresses it recursively. Section 4 defines partial and competitive equilibrium and establishes existence and uniqueness. Section 5 examines what features of the distribution of private information makes one type of asset more liquid and more expensive than another. That section also explores how a change in the extent of private information can cause a fire sale and flight-to-quality. Section 6 explores the impact of two potential policy interventions, a public asset purchase program and a subsidy to purchasing assets that decreases with the purchase price. Section 7 extends the model to have persistent preference shocks and shows that the frictions survive in the continuous time limit. Section 8 concludes.

2 Model

There is a unit measure of risk-neutral individuals. In each period t, they can be in one of two states, $s_t \in \{l, h\}$, which determines their discount factor β_{s_t} between periods t and $t+1$. We assume $0 < \beta_l < \beta_h < 1$. The preference shock is independent across individuals, which potentially allows for gains from trade. For now we assume that the preference shock is also independent over time. Thus π_s denotes the probability that an individual is in state $s \in \{l, h\}$ in any period, and it is also the fraction of individuals who are in state s in any period. For any particular individual, let $s^t \equiv \{s_0, \ldots, s_t\}$ denote the history of states through period t.

There is a finite number of different quality levels, indicated by $j \in \{1, \ldots, J\}$. Assets are perfectly durable and their supply is fixed; let K_j denote the measure of quality j assets in the economy. Each quality j asset produces δ_j units of a homogeneous, nondurable consumption good each period, and so aggregate consumption $\sum_{j=1}^{J} \delta_j K_j$ is fixed. Without loss of generality, assume that higher quality assets produce more of the consumption good, $0 \leq \delta_1 < \cdots < \delta_J$. The assumption that there is a finite number of qualities simplifies our notation and exposition. In Section 5.1 we describe the equilibrium with a continuum of assets.

We are interested in how a market economy allocates consumption across individuals. For the next three sections, we refer to the assets as "trees" and the consumption good as "fruit." The timing of events within period t is as follows:

1. each individual i owns a vector $\{k_{i,j}\}_{j=1}^{J}$ of trees which produce fruit;
2. each individual's discount factor between periods t and $t+1$ is realized;
3. individuals trade trees for fruit in a competitive market;
4. individuals consume the fruit that they hold.

We require that each individual's consumption and holdings of each quality of tree are nonnegative in every period and we do not allow any other trades, e.g. contingent claims against shocks to the discount factor. In addition, we assume that only the owner of a tree can observe its quality, creating an adverse selection problem; however, we assume that individuals' discount factors are observable. Key to our equilibrium concept, which we discuss below, is that the buyer of a tree may be able to infer its quality from the price at which it is sold.

With observable discount factors, a version the Milgrom and Stokey (1982) "no trade theorem" implies that high discount factor individuals never sell trees and low discount factor individuals never buy trees in any equilibrium despite the presence of private information.[3] For this reason, we refer to individuals with low discount factors as "sellers" and those with high discount factors as "buyers." Trade in trees for fruit therefore transfers consumption from patient individuals to impatient ones.

We now describe the market structure more precisely. After trees have borne fruit, a continuum of markets distinguished by their positive price $p \in \mathbb{R}_+$ may open up. Each buyer may take his fruit to any market (or combination of markets), attempting to purchase trees

[3]This is not necessarily true with unobservable discount factors. In the conclusion we discuss such an environment and argue that despite this, it may still be the case that in equilibrium high discount factor individuals do not want to sell trees and low discount factor individuals do not want to buy trees. Our equilibrium is therefore unaffected by this additional source of private information for an open set of parameter values.

in that market. Each seller may take his trees to any market (or combination of markets) attempting to sell trees in that market. However, each piece of fruit and each tree may only be brought to one market.

All individuals have rational beliefs about the ratio of buyers to sellers in all markets. Let $\Theta(p)$ denote the ratio of the amount of fruit brought by buyers to a market p, relative to the cost of purchasing all the trees in that market at a price p. If $\Theta(p) < 1$, there is not enough fruit to purchase all the trees offered for sale in the market, while if $\Theta(p) > 1$, there is more than enough. A seller believes that if he brings a tree to a market p, it will sell with probability $\min\{\Theta(p), 1\}$. That is, if there are excess trees in the market, the seller believes that his sale may be rationed. Likewise, a buyer who brings p units of fruit to market p believes that he will buy a tree with probability $\min\{\Theta(p)^{-1}, 1\}$. If there is excess fruit in the market, he may be rationed. A seller who is rationed keeps his tree until the following period, while a buyer who is rationed must eat his fruit.

Individuals also have rational beliefs about the quality of tree sold in each market. Let $\Gamma(p) \equiv \{\gamma_j(p)\}_{j=1}^J \in \Delta^J$ denote the probability distribution over trees available for sale in a market p, where Δ^J is the J-dimensional unit simplex.[4] Buyers expect that, conditional on buying a tree at a price p, it will be a quality j tree with probability $\gamma_j(p)$. Buyers only learn the quality of the tree that they have purchased after giving up their fruit. They have no recourse if unsatisfied with the quality.

Although trade does not happen at every price p, the functions Θ and Γ are not arbitrary. Instead, if $\Theta(p) < \infty$ (the buyer-seller ratio is finite) and $\gamma_j(p) > 0$ (a positive fraction of the trees for sale are of quality j), sellers must find it weakly optimal to sell quality j trees at price p. Without this restriction on beliefs, there would be equilibria in which, for example, no one pays a high price for a tree because everyone believes that they will only purchase low quality trees at that price. We define equilibrium precisely in Section 4 below.

We assume throughout this paper that the endogenous functions Θ and Γ are constant over time, so the environment is in a sense stationary. This restriction seems natural to us, and indeed we are able to prove existence and uniqueness of an equilibrium with this property. Key to this result is that, although the distribution of tree holdings across individuals evolves over time, the fraction of quality j trees held by individuals with a high discount factor is necessarily a constant π_h at the start of every period because preferences are independently and identically distributed over time.

[4]That is, $\gamma_j(p) \geq 0$ for all j and $\sum_{j=1}^J \gamma_j(p) = 1$.

3 Individual's Problem

Each individual starts off at time 0 with some vector of tree holdings $\{k_j\}_{j=1}^J$ and preference state $s \in \{l, h\}$. In each subsequent period t and history of preference shocks s^t, he decides how many trees to attempt to buy or sell at every possible price p, recognizing that he may be rationed at some prices and that the price may affect the quality of the trees that he buys. Let $V_s^*(\{k_j\})$ denote the supremum of the individual's expected lifetime utility over feasible policies, given initial preferences s and tree holdings $\{k_j\}$. In an online Appendix, we characterize this value explicitly and prove that it is linear in tree holdings: $V_s^*(\{k_j\}) \equiv \sum_{j=1}^J v_{s,j} k_j$ for some positive numbers $v_{s,j}$. This is a consequence of the linearity of both the individual's objective function and the constraints that he faces.

In addition, we prove that the marginal value of tree holdings satisfies relatively simple recursive problems. A seller solves

$$v_{l,j} = \delta_j + \max_{p \in \mathbb{R}_+} \big(\min\{\Theta(p), 1\} p + (1 - \min\{\Theta(p), 1\}) \beta_l \bar{v}_j \big), \tag{1}$$

where

$$\bar{v}_j \equiv \pi_h v_{h,j} + \pi_l v_{l,j}. \tag{2}$$

The individual receives a dividend of δ_j units of fruit from the tree and also gets p units of fruit if he manages to sell the tree at the chosen price p. Otherwise he keeps the tree until the following period. Note that there is no loss of generality in assuming that a seller always tries to sell all his trees, since he can always offer them at a high price such that this is optimal, $p > \beta_l \bar{v}_j$. Of course, at such a high price, he may be unable to sell it, $\Theta(p) = 0$, in which case the outcome is the same as holding onto the tree.

Similarly, a buyer solves

$$v_{h,j} = \max_{p \in \mathbb{R}_+} \left(\min\{\Theta(p)^{-1}, 1\} \frac{\delta_j}{p} \beta_h \sum_{j'} \gamma_{j'}(p) \bar{v}_{j'} + (1 - \min\{\Theta(p)^{-1}, 1\}) \delta_j \right) + \beta_h \bar{v}_j.$$

A quality j tree delivers δ_j of fruit, which the buyer uses in an attempt to purchase trees at an optimally chosen price p. If he succeeds, he buys δ_j / p trees of unknown quality, j' with probability $\gamma_{j'}(p)$, while if he fails he consumes the fruit. Finally, he gets the continuation value of the tree in the next period. Again, note that a buyer always finds it weakly optimal to attempt to purchase a tree at a sufficiently low price p, rather than simply consuming the fruit without attempting to purchase a tree. We therefore do not explicitly incorporate this last option in the value function.

Since the maximand is multiplicative in δ_j, we can equivalently write the buyer's value function as

$$v_{h,j} = \delta_j \lambda + \beta_h \bar{v}_j, \tag{3}$$

where

$$\lambda \equiv \max_p \left(\min\{\Theta(p)^{-1}, 1\} \frac{\beta_h \sum_{j=1}^J \gamma_j(p) \bar{v}_j}{p} + \left(1 - \min\{\Theta(p)^{-1}, 1\}\right) \right). \tag{4}$$

The variable λ is the endogenous value of a unit of fruit to a buyer, independent of the quality of tree that produced the fruit. If $\lambda = 1$, a unit of fruit is simply worth its consumption value, and so buyers find it weakly optimal to consume their fruit. But we may have $\lambda > 1$ in equilibrium, so buyers strictly prefer to use their fruit to purchase trees.

Proposition 1 *Let $\{v_{s,j}\}$, $\{\bar{v}_j\}$, and λ be positive-valued numbers that solve the Bellman equations* (1)–(4) *for $s = l, h$. Then $V_s^*(\{k_j\}) \equiv \sum_{j=1}^J v_{s,j} k_j$ for all $\{k_j\}$.*

The proof is in an online appendix. Note that for some choices of the functions Θ and Γ, there is no positive-valued solution to the Bellman equations. In this case, the price of trees is so low that it is possible for an individual to obtain unbounded utility and there is no solution to the individual's problem. Not surprisingly, this cannot be the case in equilibrium.

4 Equilibrium

4.1 Partial Equilibrium

We are now ready to define equilibrium. We do so in two steps. First, we define an equilibrium where the buyer's value of fruit λ is fixed, which we call "partial equilibrium". This definition is a natural dynamic extension to the definition of equilibrium in Guerrieri, Shimer and Wright (2010).[5] Then, we turn to the complete definition of a competitive equilibrium, where the value of λ is endogenous and ensures that the fruit market clears.

Definition 1 *A partial equilibrium for fixed $\lambda \geq 1$ is a pair of vectors $\{v_{h,j}\} \in \mathbb{R}_+^J$ and $\{v_{l,j}\} \in \mathbb{R}_+^J$, functions $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ and $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$, and a nondecreasing function $F : \mathbb{R}_+ \mapsto [0, 1]$ with support $\mathbb{P}$ satisfying the following conditions:*

1. *Sellers' Optimality: for all $j \in \{1, \ldots, J\}$, $v_{l,j}$ solves* (1) *where $\bar{v}_j$ is defined in* (2)*;*
2. *Equilibrium Beliefs: for all $j \in \{1, \ldots, J\}$ and for all p with $\Theta(p) < \infty$ and $\gamma_j(p) > 0$, p solves the maximization problem on the right-hand side of equation* (1)*;*

[5] The proof of existence and uniqueness of equilibrium in Guerrieri, Shimer and Wright (2010) uses a sorting assumption that is not satisfied in our environment. For that reason, we cannot directly apply our earlier proofs.

3. *Buyers' Optimality: for all* $j \in \{1, \ldots, J\}$, $v_{h,j}$ *solves* (3) *where* λ *is defined in* (4) *and* $\bar{v}_j$ *in* (2);

4. *Active Markets:* $p \in \mathbb{P}$ *only if it solves the maximization problem on the right-hand side of equation* (4);

5. *Consistency of Supply with Beliefs: for all* $j \in \{1, \ldots, J\}$,

$$\frac{K_j}{\sum_{j'} K_{j'}} = \int_{\mathbb{P}} \gamma_j(p) dF(p).$$

Sellers' Optimality requires that sellers choose an optimal price for selling each quality tree, given the ease of trade. *Equilibrium Beliefs* imposes that if individuals expect some quality j trees to be for sale at price p, it must be weakly optimal to sell quality j trees at that price.[6] *Buyers' Optimality* states that buyers choose an optimal price to buy trees, given the ease of trade and the composition of trees for sale at each price. *Active Markets* imposes that if there is trade at a price p, this must be an optimal price for buying trees. Finally, *Consistency of Supply with Beliefs* imposes that the share of sellers' trees that are of quality j is equal to the fraction of quality j trees among those offered for sale, where F denotes the fraction of trees that are offered for sale at a price less than or equal to p.

We characterize partial equilibria using a sequence of constrained optimization problems:

Definition 2 *For given* λ, *a solution to problem* (P_j) *is a vector* $(v_{l,j}, \bar{v}_j, \theta_j, p_j)$ *that solves the following Bellman equation*

$$v_{l,j} = \delta_j + \max_{p,\theta} \big(\min\{\theta, 1\} p + (1 - \min\{\theta, 1\}) \beta_l \bar{v}_j \big)$$

$$\text{s.t. } \lambda \leq \min\{\theta^{-1}, 1\} \frac{\beta_h \bar{v}_j}{p} + \big(1 - \min\{\theta^{-1}, 1\}\big), \tag{5}$$

$$\text{and } v_{l,j'} \geq \delta_{j'} + \min\{\theta, 1\} p + (1 - \min\{\theta, 1\}) \beta_l \bar{v}_{j'} \text{ for all } j' < j \tag{6}$$

with

$$\bar{v}_j = \pi_h (\delta_j \lambda + \beta_h \bar{v}_j) + \pi_l v_{l,j}.$$

We are interested in solving the sequence of problems $(P) \equiv \{(P_1), \ldots, (P_J)\}$. To do so, start with Problem (P_1). Constraint (6) disappears from Problem (P_1), and so we can solve directly for $v_{l,1}$ and $\bar{v}_1$, as well as the optimal policy p_1 and θ_1. Standard arguments ensure that the maximized value is unique if $\lambda \geq 1$. In general, for Problem (P_j), constraints (5)

[6] This condition rules out many other allocations that we are believe are unreasonable. See Guerrieri, Shimer and Wright (2010) for a further discussion of our equilibrium concept.

and (6) for $j' = j - 1$ bind, which uniquely determines p_j and θ_j as well as $v_{l,j}$ and $\bar{v}_j$ given $v_{l,j-1}$ and $\bar{v}_{j-1}$. Proceeding by induction yields the following Lemma:

Lemma 1 *For fixed $\lambda \in [1, \beta_h/\beta_l]$, the solution to the sequence of problem (P) has $v_{l,j+1} > v_{l,j}$, $\bar{v}_{j+1} > \bar{v}_j$, $p_{j+1} > p_j$, and $\theta_{j+1} \leq \min\{\theta_j, 1\}$ for all $j < J$. It is the unique such solution to the system of equations*

$$\lambda p_j = \beta_h \bar{v}_j \text{ for all } j, \tag{7}$$

$$v_{l,j} = \delta_j + \min\{\theta_j, 1\} p_j + (1 - \min\{\theta_j, 1\}) \beta_l \bar{v}_j \text{ for all } j, \tag{8}$$

$$\bar{v}_j = \pi_h(\delta_j \lambda + \beta_h \bar{v}_j) + \pi_l v_{l,j} \text{ for all } j, \tag{9}$$

$$v_{l,j-1} = \delta_{j-1} + \theta_j p_j + (1 - \theta_j) \beta_l \bar{v}_{j-1} \text{ for all } j > 1, \tag{10}$$

and $\theta_1 \geq 1$ if $\lambda = 1$, $\theta_1 \leq 1$ if $\lambda = \beta_h/\beta_l$, and $\theta_1 = 1$ otherwise.

Condition (7) is the buyer's indifference condition when the value of fruit is λ. Condition (8) is the value function of a seller and condition (9) is the value of individual before the realization of his preference shock. Condition (10) is the binding local downward incentive constraint; all other incentive constraints are slack.

If $\delta_1 > 0$ and $\lambda < \beta_h/\beta_l$, this defines $\theta_j > 0$ for all j; otherwise $\theta_j = 0$ for all $j \geq 2$. We focus on values of λ between 1 and β_h/β_l because these are the relevant ones for equilibrium. One could, however, also characterize the solution to problem (P) for $\lambda > \beta_h/\beta_l$; it would have $\theta_j = 0$ for all j.

Proposition 2 *Fix $\lambda \in [1, \beta_h/\beta_l]$. There exists a partial equilibrium. Any solution to problem (P) is a partial equilibrium and any partial equilibrium solves problem (P). More precisely:*

- *Existence: Take any $\{v_{l,j}, \bar{v}_j, \theta_j, p_j\}$ that solves problem (P). Then there exists a partial equilibrium $(\{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, F)$ where $\Theta(p_j) = \theta_j$, $\gamma_j(p_j) = 1$, $v_{h,j} = \delta_j \lambda + \beta_h \bar{v}_j$, and $dF(p_j) = K_j / \sum_{j'} K_{j'}$.*

- *Uniqueness: Take any partial equilibrium $(\{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, F)$. For all j, there exists a $p_j \in \mathbb{P}$ with $\gamma_j(p_j) > 0$. If also $\Theta(p_j) > 0$, then $(v_{l,j}, \bar{v}_j, \Theta(p_j), p_j)$ solves problem (P_j).*

The proof in the appendix gives a complete characterization of the partial equilibrium, including the entire functions Θ and Γ. Since we proved in Lemma 1 that the solution to problem (P) is unique, except possibly for the value of θ_1, this essentially proves uniqueness of the partial equilibrium.

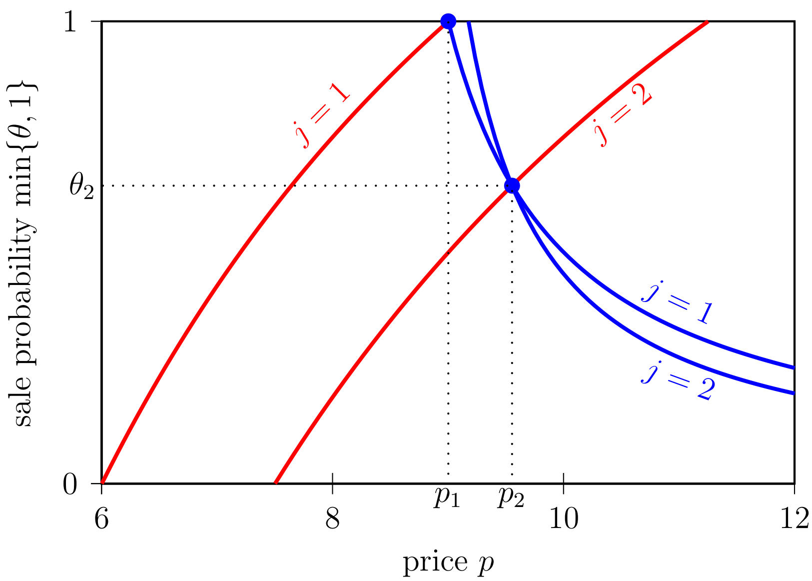

Figure 1: Illustration of problem (P) and partial equilibrium.

Figure 1 illustrates a partial equilibrium for the case with $J = 2$.[7] The two upward-sloping curves indicate pairs of prices and resale probabilities such that buyers are willing to purchase each of the trees when the value of a unit of fruit is λ. A buyer is willing to pay a higher price for a tree if he anticipates being able to resell it with a higher probability when he becomes a seller at some future date. In terms of problem (P), these curves describe the relationship between θ_j and p_j implied by equations (7)–(9) conditional on λ. This recognizes that the continuation value $\bar{v}_j$ accounts for the resaleability of the tree.

The two downward-sloping curves are the indifference curves for the seller of each of the trees evaluated at their equilibrium values. Each of them is downward sloping because a seller is willing to accept a lower sale probability if he receives a higher price conditional on a sale. The seller of tree $j = 1$ is not constrained by worse trees and so in equilibrium is able to sell the tree with probability 1. The indifference curve of this seller therefore intersects the buyers' indifference curve at a price that reflects the complete liquidity of this tree. To construct this indifference curve, first compute $\bar{v}_1$ from equations (7)–(9) and the condition $\theta_1 = 1$. Then eliminate $v_{l,1}$ from equations (8) and (10) and solve for θ_2 as a function of p_2 given this value of $\bar{v}_1$.

The seller of tree $j = 2$ is constrained by the need to signal that he holds the high quality tree. The point (p_2, θ_2) leaves the seller of a quality 1 tree indifferent between attempting

[7] The figure assumes $\beta_h = 0.9$, $\beta_l = 0.8$, $\pi_h = \pi_l = 0.5$, $\delta_1 = 1$, $\delta_2 = 1.25$, and $\lambda = 1$, an illustrative example.

to sell it for p_2 with probability θ_2 and selling it for sure at the lower price p_1. Moreover, buyers are willing to purchase quality 2 trees at price p_2 when they recognize that they can resell them with probability θ_2. Buyers would only pay a higher price for quality 2 trees if the resale probability were higher, but then the sellers of quality 1 trees would attempt to sell at this higher price.

The figure also illustrates the indifference curve of the seller of a quality 2 asset through the equilibrium price-sale probability pair (p_2, θ_2). We construct this in the same manner as a quality 1 seller's indifference curve. Note that the sellers' indifference curves satisfy a single-crossing property, which is key to our separating equilibrium. The owner of a higher quality tree is willing to accept a greater reduction in the sale probability for a given increase in the price because the continuation value of holding a higher quality tree is higher. This illustrates how higher quality trees sell at a higher price but with a lower probability in equilibrium. Finally, if there are more types of trees, we can use a similar inductive procedure to construct the price and sale probability for each quality level.

4.2 Competitive Equilibrium

We now turn to a full competitive equilibrium in which λ is endogenous:

Definition 3 *A competitive equilibrium is a number $\lambda \in [1, \beta_h/\beta_l]$, a pair of vectors $\{v_{h,j}\} \in \mathbb{R}_+^J$ and $\{v_{l,j}\} \in \mathbb{R}_+^J$, functions $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ and $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$, and a nondecreasing function $F : \mathbb{R}_+ \mapsto [0, 1]$ with support $\mathbb{P}$ satisfying the following conditions:*

1. *$(\{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, F)$ is a partial equilibrium for fixed λ; and*
2. *the fruit market clears: $\pi_h \sum_{j=1}^{J} \delta_j K_j = \pi_l \left(\sum_{j=1}^{J} K_j \right) \int_{\mathbb{P}} \Theta(p) p dF(p)$.*

A competitive equilibrium is a partial equilibrium plus the market clearing condition that states that the fruit brought to market by buyers is equal to the value of trees brought to the market by sellers times the buyer-seller ratio. Recall from Proposition 2 that $dF(p_j) = K_j / \sum_{j'} K_{j'}$ in partial equilibrium, where p_j is the equilibrium price of quality j trees. The market clearing condition therefore reduces to

$$\pi_h \sum_{j=1}^{J} \delta_j K_j = \pi_l \sum_{j=1}^{J} \Theta(p_j) p_j K_j. \tag{11}$$

The left hand side is the fruit held by buyers at the start of the period, while each term in the right hand side is the equilibrium cost of purchasing a particular quality tree multiplied by the buyer-seller ratio for that tree.

Proposition 3 *A competitive equilibrium* $(\lambda, \{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, F)$ *exists and is unique. For fixed values of the other parameters, there exists thresholds* $\underline{\pi} < \bar{\pi}$ *such that*

$$\pi_l \begin{cases} \geq \bar{\pi} & \lambda = \beta_h/\beta_l \\ \in (\underline{\pi}, \bar{\pi}) \Rightarrow & \lambda \in (1, \beta_h/\beta_l) \\ \leq \underline{\pi} & \lambda = 1. \end{cases}$$

The proof shows that an increase in the value of fruit to a buyer λ drives down the amount of fruit that sellers expect to get from selling any quality j tree, that is, $p_j\Theta(p_j)$. Indeed, in the limit when $\lambda = \beta_h/\beta_l$, $\Theta(p_j) = 0$ for all $j > 1$, and so trade breaks down in all but the worst quality tree. At the opposite limit of $\lambda = 1$, buyers are indifferent about purchasing trees and so $\Theta(p_1) > 1$ and buyers are rationed. By varying λ, we find the unique value at which the fruit market clears.

The proposition also shows that the value of fruit is increasing in the fraction of sellers. When $\pi_l \geq \bar{\pi}$, the unique equilibrium has $\lambda = \beta_h/\beta_l$, the abundance of sellers drives down the price of trees. In equilibrium, there is only a market in the lowest quality tree and sellers are indifferent about selling that tree. When $\pi_l \leq \underline{\pi}$, a shortage of sellers drives up the price of trees until the point where buyers are indifferent about consuming their fruit. At intermediate values of π_l, $\beta_h/\beta_l > \lambda > 1$, there is a market for every quality tree, and buyers use all their fruit to purchase trees. The thresholds satisfy $1 > \bar{\pi} > \underline{\pi} > 0$ and depend on all the other model parameters.

5 Positive Implications

We now use our model to explore the link between asymmetric information, asset prices, and liquidity. We show how private information about the quality of an asset can reduce the amount of the asset that is traded each period, its liquidity, as well as the price-dividend ratio of the asset, which we label a fire sale. The emergence of private information about one asset can therefore raise the price of other safe assets through general equilibrium effects, a flight-to-quality.

To discuss these issues, we find it useful to extend our environment along two dimensions: we develop a version of our model in which there are many types of assets and a continuum of quality levels for each type of asset. We assume that buyers are able to observe an asset's type but not its quality level. The assumption that there are many types of assets is useful for exploring how private information about an asset's quality affects its price and liquidity, and for examining how a change in the extent of private information affects the price and liquidity of other assets. The assumption that there is a continuum of quality levels for each type of asset is useful for obtaining closed-form solutions for some key expressions, which we use in our proofs.

5.1 Two Extensions

We start by formally introducing our two extensions. First, we assume that there are A *types* of assets (or more concisely, A assets), named $a = 1, \ldots, A$. All individuals can observe an asset's type, but only an asset's owner knows its *quality*, the dividend that the asset produces in each period. The dividends produced by different types and qualities of assets are perfect substitutes in consumption, so the only practical difference between different types of assets lies in the extent of the asymmetric information problem. Let K_a denote the measure of type a assets in the economy.

Second, we let $G_a(\delta)$ denote the cumulative distribution of dividends, i.e. the quality distribution, among type a assets and assume that the support of G_a is the convex interval $[\underline{\delta}_a, \bar{\delta}_a]$, continuous rather than finite. We also let g_a denote the associated density when it exists. If $\underline{\delta}_a = \bar{\delta}_a$, then the asset is *safe*, since the buyer knows the asset's quality δ by observing its type a. Otherwise asymmetric information plays a role in pricing the asset.

The definitions of partial and competitive equilibria are natural extensions of Definitions 1 and 3. For example, the rationing function Θ_a and belief function Γ_a are now defined conditional on the asset's type a. In addition, the belief function is now a probability distribution function defined on the interval $[\underline{\delta}_a, \bar{\delta}_a]$ for each price p. We omit the relevant Bellman equations and formal definitions in the interest of space.[8]

We can prove that for any value of $\lambda \in [1, \beta_h/\beta_l]$, there exists a unique partial equilibrium in which the price-dividend ratio for any asset reflects its liquidity, with the price $P_a(\delta)$ solving

$$P_a(\delta) = \beta_h \frac{\delta(\pi_l + \lambda\pi_h)}{\lambda(1 - \pi_l\beta_l - \pi_h\beta_h) - \pi_l(\beta_h - \beta_l\lambda)\Theta_a(P_a(\delta))} \tag{12}$$

increasing in δ. This equation follows immediately from Bellman equations analogous to equations (7)–(9) and ensures that buyers are willing to purchase type a, quality δ assets when the value of consumption is λ. The lowest quality type a asset sells with probability 1, $\Theta_a(P_a(\underline{\delta})) = 1$, defining a minimum price $\underline{p}_a \equiv P_a(\underline{\delta})$ which we can find immediately from

[8] One minor modification is the market clearing condition when $\lambda = 1$ or $\lambda = \beta_h/\beta_l$. In the economy with finitely many quality levels, we used $\Theta(p_1) \geq 1$ to ensure that buyers brought all their dividends to the market even when $\lambda = 1$. Here it is easier to allow buyers to consume a positive fraction of their dividends and impose $\Theta(P(\underline{\delta})) = 1$. Similarly, if $\lambda = \beta_h/\beta_l$, we had previously imposed $\Theta(p_1) \leq 1$. Here it is easier to assume that sellers do not necessarily sell all of their worst assets.

equation (12). Finally, the sale probability of a type a asset at price p is

$$\Theta_a(p) = \left(\underline{p}_a/p\right)^{\frac{\beta_h}{\beta_h - \beta_l \lambda}}. \tag{13}$$

Note that if $\lambda = \beta_h/\beta_l$, this implies $\Theta_a(p) = 0$ whenever $p > \underline{p}_a$, while otherwise $\Theta_a(p)$ is strictly decreasing. Equation (13) ensures that a seller holding an asset with dividend δ prefers to sell it at the price $P_a(\delta)$, consistent with equilibrium beliefs.[9]

These equations hold if $\delta \in [\underline{\delta}_a, \bar{\delta}_a]$ and so $p = P_a(\delta) \in [\underline{p}_a, \bar{p}_a]$. At lower prices $p < \underline{p}_a$, $\Theta_a(p) = \infty$ and the asset quality is arbitrary. At higher prices $p > \bar{p}_a$, $\Theta_a(p)$ is pinned down by the indifference curve of the seller of a quality $\bar{\delta}_a$ asset and buyers believe that they will get an asset with quality $\bar{\delta}_a$ at these prices. Using arguments similar to the ones we develop in a recent working paper, Guerrieri and Shimer (2013), we can prove that this defines the unique partial equilibrium.[10]

To construct a competitive equilibrium, we find the value of consumption good λ that clears the market,

$$\pi_h \sum_{a=1}^{A} K_a \int_{\underline{\delta}_a}^{\bar{\delta}_a} \delta dG_a(\delta) \gtreqless \pi_l \sum_{a=1}^{A} K_a \int_{\underline{\delta}_a}^{\bar{\delta}_a} \Theta_a(P_a(\delta)) P_a(\delta) dG_a(\delta) \Leftrightarrow \lambda \begin{cases} = 1 \\ \in [1, \beta_h/\beta_l] \\ = \beta_h/\beta_l. \end{cases} \tag{14}$$

If the left hand side exceeds the right hand side, the difference is the measure of the consumption good consumed by buyers. If the right hand side exceeds the left, the difference represents the value of the lowest quality assets not offered for sale by sellers.[11] We can prove directly from the functional forms of Θ and P that an increase in λ reduces the right hand side of this inequality, ensuring that the competitive equilibrium is unique.

5.2 Liquidity, Volume, and Price of Different Assets

We start our analysis by asking what properties of an asset make it liquid, lead to a high volume, and lead to high asking and transactions prices. We define these concepts as follows:

[9] Sellers choose the price p to maximize $\Theta_a(p)(p - \beta_l \bar{v}_a(\delta))$, where $\bar{v}_a(\delta)$ is the continuation value of holding a type a asset with quality δ. The first order condition from this problem is $\Theta_a(p) + \Theta_a'(p)(p - \beta_l \bar{v}_a(\delta)) = 0$. With this functional form, $\Theta_a'(p) = -\frac{\beta_h}{\beta_h - \beta_l \lambda}\Theta_a(p)/p$ and so this reduces to $\lambda P_a(\delta) = \beta_h \bar{v}_a(\delta))$, the analog of equation (7), which ensures that buyers are willing to purchase the asset when the value of the consumption good is λ.

[10] An alternative approach that we do not pursue here is to consider the economy with a continuum of asset qualities as the limit of a sequence of economies with increasingly many quality levels.

[11] Recall that when $\lambda = \beta_h/\beta_l$, $\Theta_a(p) = 0$ if $p > \underline{p}_a$.

Definition 4 *The* liquidity *of asset a is the fraction of the asset sold in each period:*

$$L_a \equiv \pi_l \int_{\underline{\delta}_a}^{\bar{\delta}_a} \Theta_a(P_a(\delta)) dG_a(\delta).$$

The volume *of asset a is the amount of the consumption good exchanged for the asset in each period:*

$$V_a \equiv \pi_l \int_{\underline{\delta}_a}^{\bar{\delta}_a} \Theta_a(P_a(\delta)) P_a(\delta) dG_a(\delta).$$

The average asking price *of asset a is the unweighted average price of the asset:*

$$A_a \equiv \int_{\underline{\delta}_a}^{\bar{\delta}_a} P_a(\delta) dG_a(\delta).$$

The average transaction price *of asset a is the price weighted by sale probabilities:*

$$T_a \equiv \frac{\int_{\underline{\delta}_a}^{\bar{\delta}_a} \Theta_a(P_a(\delta)) P_a(\delta) dG_a(\delta)}{\int_{\underline{\delta}_a}^{\bar{\delta}_a} \Theta_a(P_a(\delta)) dG_a(\delta)}.$$

We focus throughout this section on parameter values for which $1 \leq \lambda < \beta_h/\beta_l$. If $\lambda = \beta_h/\beta_l$, sellers are indifferent about selling any of their assets and so the above definitions of liquidity and volume need not apply.[12]

We first show that if one asset produces κ times the dividends of another asset, prices are proportional and liquidity is the same for the two assets.

Proposition 4 *Assume $1 \leq \lambda < \beta_h/\beta_l$. Consider any two assets with distributions satisfying $G_1(\kappa\delta) = G_2(\delta)$ for all δ and for some $\kappa > 0$, so in particular $\underline{\delta}_1 = \kappa\underline{\delta}_2$. Then $P_1(\kappa\delta) = \kappa P_2(\delta)$ and $\Theta_1(\kappa p) = \Theta_2(p)$ for all p. In particular, the two assets have the same liquidity, $L_1 = L_2$, while volume, asking price, and transaction price are proportional, $V_1 = \kappa V_2$, $A_1 = \kappa A_2$, and $T_1 = \kappa T_2$.*

This claim can be verified directly using equations (12) and (13). The most important component of this result is that an asset's liquidity is not related to whether it produces a high dividend, but instead depends on the second moment of the dividend. We turn to this relationship next.

We start by showing a particular sense in which an increase in the precision of a seller's information about asset quality reduces all four of these concepts. Take two assets and assume that asset 2 is riskier than asset 1 in the following sense: the quality of asset 2 is the

[12]In particular, $\Theta_a(p) = 0$ if $p > \underline{p}_a$.

product of quality of asset 1 and an independent, nonnegative random variable with mean 1. That is, the densities satisfy

$$g_2(\delta) = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} h(\varepsilon) g_1(\delta/\varepsilon) d\varepsilon, \tag{15}$$

where h is a random variable with mean 1 and support $[\underline{\varepsilon}, \bar{\varepsilon}]$. For example, suppose asset 1 is a safe asset with dividend $\underline{\delta}_1 = \bar{\delta}_1 = \delta_1$, Then $g_2(\delta) = h(\delta/\delta_1)$, so asset 2 is a risky asset with the same mean as asset 1 but a positive variance. Alternatively, suppose the true quality of both types of assets is the product of two components, δ and ε, drawn independently from the densities g_1 and h respectively. The owners of asset 1 are only able to observe δ (and, to keep with the structure of our model, do not observe the dividend and so cannot infer ε), while the owners of asset 2 can observe both components, and in particular the density g_2. Then sellers of asset 2 have more precise information than sellers of asset 1. Superior information of this type reduces both asset prices and liquidity:

Proposition 5 *Assume* $1 \leq \lambda < \beta_h/\beta_l$. *Consider two assets with distributions satisfying equation* (15). *Then asset* 1 *is more liquid, has a higher volume, has a higher average asking price, and has a higher average transaction price than asset* 2.

Despite this proposition, assets with more disperse distributions and the same mean need not have lower prices and liquidity. A useful counterexample is the case in which two assets have the same support $[\underline{\delta}, \bar{\delta}]$, but one distribution second order stochastically dominates the other. For example, suppose there are two types of assets and half of all assets are "bad" while the other half are "good." A bad asset has a dividend of $\underline{\delta}$ and a good asset has a dividend of $\bar{\delta}$. The owners of asset 1 initially observe one signal about the quality of the asset (and again do not later observe the dividend and so cannot update this signal). The signal is informative about the quality of the asset and may be arbitrarily accurate. Using Bayes law, it is straightforward to compute the cumulative distribution of the expected dividend conditional on the single signal, $G_1(\delta)$ with support $[\underline{\delta}, \bar{\delta}]$. Asset 2 is identical, except that the owners of asset 2 initially get two independent signals drawn from the same distribution. Their posterior belief about the quality of the asset is some $G_2(\delta)$ with support $[\underline{\delta}, \bar{\delta}]$. An extra signal moves mass towards the extremes, and so the cumulative distribution G_1 second order stochastically dominates G_2. In this case, a more accurate signal actually raises liquidity, volume and average asking price:

Proposition 6 *Assume* $1 \leq \lambda < \beta_h/\beta_l$. *Consider two types of assets with the same support,* $\underline{\delta}_1 = \underline{\delta}_2$ *and* $\bar{\delta}_1 = \bar{\delta}_2$ *and assume that the cumulative distribution function* G_1 *second order* *stochastically dominates G_2. Then asset 1 is less liquid, has a lower volume, and has a lower average asking price than asset 2.*

In general asset 1 need not have a lower average transaction price under these conditions.

The contrasting results in Propositions 5 and 6 indicate two opposing forces from an increase in the dispersion of an asset's quality. On the one hand, a reduction in the lower bound of the support of the quality distribution increases trading frictions and reduces prices. On the other hand, holding fixed the support of the distribution, liquidity, volume, and asking prices are convex functions of quality and so on average rise with dispersion. In general, we expect both of these forces to be at work and so cannot predict whether an increase in the extent of asymmetric information for one asset compared to another will raise or lower liquidity, volume, and prices without understanding the exact nature of the change. In this sense, our theory warns against naïve but perhaps ex ante plausible predictions, such as that an increase in sellers' private information will always reduce liquidity, volume, and prices. This is true only if there is a sufficient decline in the lower bound of the support of the quality distribution. And of course both of these propositions are contingent on the assets' average dividends; if that differs as well, as may be reasonable in many applications, tthen the results in Proposition 4 are also relevant.

5.3 Fire Sales, Liquidity, and Flight to Quality

Define a *fire sale* for an asset as a decline in its average asking price and a *flight-to-quality* as a simultaneous decrease in the volume of trade for that asset and increase in the volume of trade and price for other safer assets. This section explores how a change in the extent of private information about one type of asset's quality can cause a fire sale for that asset, a collapse in its liquidity, and consequently a flight-to-quality in a competitive equilibrium.

To be concrete, we assume without loss of generality that the quality distribution of type 1 assets changes from G_1 to $\tilde{G}_1$, with other distributions unchanged. We only consider a one time, permanent, and unanticipated change in the distribution, although it would be straightforward to develop a version of the model in which agents anticipate that the quality distribution may change in the future. Since the model has no payoff-relevant state variable, we accordingly perform a comparative statics exercise.

First consider a partial equilibrium exercise in which the value of dividends to buyers, λ, is held fixed. Then the impact of a change in the asset's quality is completely described by the analysis in the previous section. For example, if the dividend is subjected to a multiplicative shock, liquidity, volume, average asking price, and average transaction price all decline (Proposition 5). This might be the most natural assumption if the asset's payoff is hit by an idiosyncratic shock. Alternatively, if the dividend's distribution spreads out in the sense of second order stochastic dominance but the support of the distribution does not change, liquidity, volume, and average asking price rise (Proposition 6). This might be the most natural assumption if sellers receive more private information about the dividend. Our model again suggests that an adverse shock might correspond to either an increase or decrease in variance of dividends, depending on the nature of shock.

In any case, a change in the distribution G_1 also has a general equilibrium effect through the value of dividends λ. The following lemma explains how a change in λ affects the equilibrium.

Lemma 2 *Compare the partial equilibrium of two economies which differ only in the value of λ. Then the liquidity, volume, average asking price, and average transaction price are all higher for all assets in the economy with a lower value of λ.*

The proof is based on differentiating the price function $P_a(\delta)$ and liquidity function $\Theta_a(P_a(\delta))$ with respect to λ.

Combining Lemma 2 with the market clearing condition (14), it is straightforward to compute the impact of a change in the distribution of dividends of one asset which affects the volume of that asset on the liquidity, volume, average asking price, and average transaction price of any other asset:

Proposition 7 *Consider any change in the distribution of quality for asset 1, from G_1 to $\tilde{G}_1$ with $\int \delta dG_1(\delta) = \int \delta d\tilde{G}_1(\delta)$, that reduces the volume V_1 holding λ fixed. If initially $1 < \lambda \leq \beta_h/\beta_l$, then in a competitive equilibrium, the liquidity, volume, average asking price, and average transaction price of all other types of assets weakly increase, and strictly if $\lambda < \beta_h/\beta_l$. If initially $\lambda = 1$, then the liquidity, volume, average asking price, and average transaction price of all other types of assets is unchanged.*

Proof. By assumption the left hand side of condition (14) is unchanged while the right hand side, $\pi_l \sum_{a=1}^{A} K_a V_a$, declines at the initial value of λ. If $1 < \lambda < \beta_h/\beta_l$, this is no longer an equilibrium, while if $\lambda = 1$ it is and if $\lambda = \beta_h/\beta_l$ it may be. If it is still an equilibrium, then λ does not change. If not, λ rises to restore equilibrium, as Lemma 2 implies that a reduction in λ raises V_a for all a. Moreover, it implies that this reduction in λ is associated with an increase in the liquidity, volume, average asking price, and average transaction price for all other assets. ■

Combining Propositions 4 and 7 immediately implies:

Corollary 1 *Suppose the distribution of quality for asset 1 changes from G_1 to $\tilde{G}_1$ where $\tilde{G}_1(\delta) = G_1(\kappa\delta)$ for all δ and for some $\kappa > 1$. If initially $1 < \lambda < \beta_h/\beta_l$, then in a competitive* *equilibrium, the liquidity, volume, average asking price, and average transaction price of all other types of assets weakly increase. If initially* $\lambda = 1$*, then the liquidity, volume, average asking price, and average transaction price of all other types of assets is unchanged.*

Similarly, combining Propositions 5 and 7 implies

Corollary 2 *Suppose the density of quality for asset* 1 *changes from* g_1 *to* $\tilde{g}_1$ *where* $\tilde{g}_1(\delta) = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} h(\varepsilon) g_1(\delta/\varepsilon) d\varepsilon$ *for all* δ *and for some density* $h(\varepsilon)$ *with* $\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \varepsilon h(\varepsilon) d\varepsilon = 1$*. If initially* $1 < \lambda \leq \beta_h/\beta_l$*, then in a competitive equilibrium, the liquidity, volume, average asking price, and average transaction price of all other types of assets weakly increase. If initially* $\lambda = 1$*, then the liquidity, volume, average asking price, and average transaction price of all other types of assets is unchanged.*

And combining Propositions 6 and 7 implies

Corollary 3 *Suppose the density of quality for asset* 1 *changes from* g_1 *to* $\tilde{g}_1$*, leaving the support* $[\underline{\delta}_1, \bar{\delta}_1]$ *and the expected value* $\int_{\underline{\delta}}^{\bar{\delta}} \delta g_1(\delta) d\delta = \int_{\underline{\delta}}^{\bar{\delta}} \delta \tilde{g}_1(\delta) d\delta$ *unchanged. Assume* $\tilde{g}_1$ *second order stochastically dominates* g_1*. If initially* $1 < \lambda \leq \beta_h/\beta_l$*, then in a competitive equilibrium, the liquidity, volume, average asking price, and average transaction price of all other types of assets weakly increase. If initially* $\lambda = 1$*, then the liquidity, volume, average asking price, and average transaction price of all other types of assets is unchanged.*

The market clearing condition implies that in all of these cases, the volume of asset 1, the asset hit by the shock, must decline; however, the behavior of its price is in general ambiguous, with partial and general equilibrium effects working in opposite directions. The direct effect of the shock lowers the average asking price of asset 1, but the equilibrium impact through λ raises it. In principle, the equilibrium effect may dominate the direct effect. Nevertheless, the smaller is the share of the consumption good coming from asset 1,

$$\frac{K_1 \int_{\underline{\delta}_1}^{\bar{\delta}_1} \delta dG_1(\delta)}{\sum_{a=1}^{A} K_a \int_{\underline{\delta}_a}^{\bar{\delta}_a} \delta dG_a(\delta)},$$

the smaller is the equilibrium effect and so the more relevant are the partial equilibrium results in Propositions 4, 5 and 6.

We view the unanticipated shocks described in these corollaries as a fire sale that induces a flight-to-quality. Our model offers a simple explanation for how a shock that adversely affects the volume in one market increases the volume, liquidity, and price in all other markets: it is a general equilibrium effect. First, any shock that lowers the price and liquidity of one asset will also lower the volume in that market. Some of the dividends that used to go towards purchasing asset 1 will instead be used to purchase alternative assets, particularly those that are good substitutes.[13] For example, a collapse in the market for bonds securitized by mortgages will raise the price of other bonds, such as sovereign debt.

We close this section with a brief discussion of the welfare consequences of a fire sale and flight-to-quality episode. In general, there are winners and losers from a flight-to-quality. Individuals who are primarily holding assets other than asset 1 gain, since their assets increase in price and liquidity. For example, an investor holding U.S. sovereign debt at the start of the financial crisis benefited from the fire sale in other asset markets. On the other hand, there are typically some other investors who are made worse off, particularly those trying to sell asset 1 in the midst of a fire sale. Of course, since we study an endowment economy, we cannot examine whether a fire sale and flight-to-quality causes a collapse in production, but we believe that this is likely to be the case if asset sales are needed to facilitate production.

6 Policy Implications

We believe our model may be useful for understanding the potential impact of an asset purchases and subsidies, such as the original vision for the Troubled Asset Relief Program in 2008 or the Public-Private Investment Program for Legacy Assets in 2009. These programs were designed to alleviate the adverse selection problem in the market for troubled assets, thereby improving also the solvency of institutions exposed to these assets. This was supposed to occur not only because of the direct subsidy but also through the effect on the price and liquidity of assets that were not sold to the government.

We show that these predictions are consistent with our model. We analyze two policies. The first is an asset purchase program. A large actor, say the government, pledges to purchase any amount of asset a at a price $\hat{p} > \underline{p}_a$. The second is an asset subsidy program. Now the government pledges to pay $\sigma(p)$ to any seller who sells asset a at price p. We view both of these policies as feasible, in the sense that they do not give the government information that is unavailable to other buyers. We show that although both policies are costly, they raise the liquidity and price of asset a in partial equilibrium. This result goes through for the general equilibrium when parameters are such that $\lambda = 1$ before the intervention, while it may be dampened when $\lambda > 1$.

[13] Our model in fact predicts that the price of all assets will increase, although this is likely to be an artefact of our convenient assumption that all assets are perfectly substitutable. Extending our model to allow for assets that are imperfect substitutes, for example stocks versus bonds, goes beyond the scope of this paper.

6.1 Asset Purchase Program

We analyze a one-time, unexpected, permanent intervention in the market. The government pledges to purchase any type a asset from any seller (individual with a low discount factor) at a price $\hat{p}$. It also commits never to resell those assets. We are interested in understanding the impact of such an *asset purchase program.* For the rest of this section, we drop the subscript a and focus our attention on the relevant asset.

A naïve guess is that any quality δ asset with pre-intervention price $P(\delta) < \hat{p}$ would be sold to the government, leaving the remaining assets to circulate in the private market. But if this were an equilibrium, it would become common knowledge that there is no risk of purchasing a type a asset with quality less than $P^{-1}(\hat{p})$ in the private market, effectively changing the lower bound on asset quality. This in turn would boost the liquidity and hence the price of all assets. And the higher prices would induce some holders of assets with quality less than $P^{-1}(\hat{p})$ to sell to the private sector rather than the government. Instead, the impact of an asset purchase program is as follows:

Proposition 8 *Consider an asset purchase program at price $\hat{p} > \underline{p}$. In partial equilibrium with fixed λ, any asset with quality*

$$\delta < \hat{\delta} \equiv \frac{\hat{p}\big(\lambda - \beta_h(\pi_l + \lambda\pi_h)\big)}{\beta_h(\pi_l + \lambda\pi_h)}$$

is sold to the government. Any quality $\delta \geq \hat{\delta} > \underline{\delta}$ sells at a price $\hat{P}(\delta)$ with probability $\hat{\Theta}(\hat{P}(\delta))$ satisfying

$$\hat{P}(\delta) = \beta_h \frac{\delta(\pi_l + \lambda\pi_h)}{\lambda(1 - \pi_l\beta_l - \pi_h\beta_h) - \pi_l(\beta_h - \beta_l\lambda)\hat{\Theta}(\hat{P}(\delta))} \text{ and } \hat{\Theta}(p) = (\hat{p}/p)^{\frac{\beta_h}{\beta_h - \beta_l\lambda}}.$$

The price and liquidity of all assets with quality $\delta \geq \hat{\delta}$ increases: $\hat{P}(\delta) > P(\delta)$ and $\hat{\Theta}(\hat{P}(\delta)) > \Theta(P(\delta))$.

The proposition states that any asset that is worth less than $\hat{p}$ even if it could be resold with probability 1 is sold to the government, while all other assets remain in private hands. This effectively raises the lower bound on the quality the asset from $\underline{\delta}$ to $\hat{\delta}$, boosting the price and liquidity of all remaining assets.

We next turn to the implications of the policy for our four aggregate measures, liquidity, volume, average asking price, and average transaction price. After the policy intervention, we define these measures only for the assets sold in the private market, those with quality $\delta \geq \hat{\delta}$. For example, we let $\hat{L}$ denote the fraction of the assets with $\delta \geq \hat{\delta}$ that are sold in each period, and similarly for $\hat{V}$, $\hat{A}$, and $\hat{T}$. These seem like the relevant empirical measures. Once the government has purchased all the assets with $\delta < \hat{\delta}$, the liquidity of the market is the fraction of privately-held assets sold each period, the volume is the amount of the consumption that private agents use to purchase the asset, and the asking and transaction prices are the unweighted and weighted average price of privately-held assets.

Proposition 9 *Consider an asset purchase program at price $\hat{p} > \underline{p}$. In partial equilibrium with fixed $\lambda < \beta_h/\beta_l$, the average asking price and average transaction price increase: $\hat{A} > A$ and $\hat{T} > T$. If in addition $\frac{\partial^2 \log(1-G(\delta))}{\partial(\log\delta)^2} \leq 0$ and $\bar{\delta} = \infty$, then liquidity and volume increase, $\hat{L} > L$ and $\hat{V} > V$.*

The condition for liquidity and volume to increase places the Pareto distribution $G(\delta) = 1 - (\delta/\underline{\delta})^{-\kappa}$ at the boundary. This condition is necessary for liquidity, in the sense that if $\frac{\partial^2 \log(1-G(\delta))}{\partial(\log\delta)^2} \geq 0$, our proof establishes that liquidity falls, $\hat{L} < L$. The condition for volume to increase could be weaker, but some condition is needed, as a simple counterexample shows.[14]

An asset purchase program will also have general equilibrium effects on the price and liquidity of the purchased asset a and of all other assets. The exact nature of these effects depends on other details of the program. For example, what the government does with the dividends produced by assets it purchases in the program affects the left hand side of condition (14) and hence affects the equilibrium value of dividends λ. Nevertheless, the logic behind Lemma 2 and Proposition 7 implies that if the intervention in the market for asset a raises the volume of asset a and the government does not correspondingly increase the amount of dividends used to purchase assets, then λ must increase to restore equilibrium. This then lowers the liquidity, volume, and prices for all other assets and moderates the direct impact of the intervention on asset a.

6.2 Asset Subsidy Program

We next consider a subsidy program. We assume that whenever an individual purchases a unit of asset a at a price p, the seller is paid an addition $\sigma_a(p) \geq 0$ units of the consumption

[14] We choose parameters to keep the algebra simple. Let $\lambda = 1$, $\beta_h = 0.8$, $\beta_l = 0.4$, and $\pi_h = 0.9$. Then

$$\Theta(\delta)P(\delta) = \frac{48\hat{\delta}}{5(\delta/\hat{\delta}) + \sqrt{24 + 25(\delta/\hat{\delta})^2}}.$$

Assume $g(\delta) = \frac{11}{2}(2-\delta)^{10}$ with $\underline{\delta} = 1$ and $\bar{\delta} = 3$. This distribution fails our concavity condition for $\delta \in [1, 2)$ and satisfies it for $\delta \in (2, 3]$. We find that $V = 2.69$ but then steadily decreases as $\hat{\delta}$ increases. An intervention that sets $\hat{\delta} = 1.15$ achieves the lowest volume, $\hat{V} = 2.43$.

good. We assume that the subsidy is decreasing, $\sigma_a'(p) \leq 0$, but that the seller's take home income, $p + \sigma_a(p)$ is increasing in the sale price, that is, $\sigma_a'(p) > -1$. We again drop the subscript a from our analysis for notational simplicity.

The definition of equilibrium is unchanged except for the introduction of the subsidy scheme in the analog of the sellers' Bellman equation (1) for continuous types, that is,

$$v_l(\delta) = \delta + \max_{p \in \mathbb{R}_+} \left(\min\{\hat{\Theta}(p), 1\}(p + \sigma(p)) + (1 - \min\{\hat{\Theta}(p), 1\})\beta_l \bar{v}(\delta) \right).$$

In particular, sellers still set optimal prices for their assets given the sale probability $\Theta(p)$, internalizing the fact that they get a subsidy $\sigma(p)$ if they sell an asset for p. Using this, we can prove that the price of any asset satisfies

$$\hat{P}(\delta) = \beta_h \frac{\delta(\pi_l + \lambda\pi_h) + \pi_l \hat{\Theta}(\hat{P}(\delta))\sigma(\hat{P}(\delta))}{\lambda(1 - \pi_l\beta_l - \pi_h\beta_h) - \pi_l(\beta_h - \beta_l\lambda)\hat{\Theta}(\hat{P}(\delta))}, \tag{16}$$

while the liquidity solves the differential equation

$$\frac{\hat{\Theta}'(p)}{\hat{\Theta}(p)} = \frac{-\beta_h(1 + \sigma'(p))}{p(\beta_h - \lambda\beta_l) + \beta_h\sigma(p)}. \tag{17}$$

with $\hat{\Theta}(\hat{P}(\underline{\delta})) = 1$. We omit the proof of this claim, which is unchanged from the economy without a subsidy. The price of each asset is a natural extension of equation (12) and is again obtained by combining the analogs of equations (7)–(9). The optimal sale price satisfies the seller's first order condition for maximizing problem (6.2). With an arbitrary subsidy program, we cannot solve explicitly for $\hat{\Theta}(p)$, however.

We focus on the impact of an asset subsidy program on the price and sale probability of the subsidized asset:

Proposition 10 *Consider an asset subsidy program $\sigma(p) \geq 0$ with $0 \geq \sigma'(p) > -1$. In partial equilibrium with fixed λ, the program raises the price of all assets, $\hat{P}(\delta) > P(\delta)$. In addition, it raises the sale probability of all assets, and proportionately more so for higher quality assets, $\hat{\Theta}(\hat{P}(\delta))/\Theta(P(\delta)) \geq 1$ and is increasing in δ.*

Intuitively, a subsidy to selling an asset raises the value of the asset to a buyer and therefore raises its equilibrium price. A decreasing subsidy schedule increases the sale probability of all assets by making it less costly to separate different quality assets: sellers are more willing to set a low price for a low quality asset when the relative subsidy is higher. More surprising is that, although the subsidy schedule is decreasing and the subsidy may even fall to zero, the relative sale probability increases more for the highest quality assets.

Proposition 11 *Consider an asset subsidy program $\sigma(p) \geq 0$ for asset a with $0 \geq \sigma'(p) > -1$. In partial equilibrium with fixed λ, the program raises liquidity, volume, average asking price, and average transaction price.*

This result follows almost immediately from the previous proposition.

Our analysis so far focused on a partial equilibrium for fixed λ. Clearly, the subsidy program may have general equilibrium effects. Their exact nature will depend on the details of the program, in particular on how taxes are raised to finance the subsidies. For example, a tax on individuals with high discount factors will make the consumption good scarce and so raise its value, while a tax on those with low discount factors will have the opposite impact. For similar reasons, we sidestep any discussion of welfare, since the winners and losers from any program will depend on who pays for it.

7 Persistent Shocks and Continuous Time

Our model explains how adverse selection can generate illiquid assets that only sell with a certain probability each period. But suppose that the time between periods is negligible. Does the illiquidity become negligible as well? We argue in this section that it does not. Instead, equilibrium requires that a real amount of calendar time elapse before a high quality asset is sold.

To show this, we return to the economy with one type of asset and a finite number of quality levels δ_j. We consider the behavior of the economy when the number of periods per unit of calendar time increases without bound. That is, we take the limit of the economy as the discount factors converge to 1, holding fixed the ratio of discount rates $(1-\beta_h)/(1-\beta_l)$ and the present value of dividends $\delta_j/(1-\beta_s)$. But as we take this limit, we also want to avoid changing the stochastic process of shocks. With i.i.d. shocks and very short time periods, there is almost no difference in preferences across individuals and so the gains from trade become negligible. We therefore first introduce persistent shocks into the model and then prove that as the period length shortens, the probability of sale per period falls to zero, while the probability of sale per unit of calendar time converges to a well-behaved number.

7.1 Persistent Shocks

Assume now that $s_t \in \{l, h\}$ follows a first order stochastic Markov process and let $\pi_{ss'}$ denote the probability that the state next period is s' given that the current state is s. A partial equilibrium with a fixed value of $\lambda \geq 1$ is still characterized by a pair of functions $\{v_{s,j}\} \in \mathbb{R}_+^{2J}$ that represent the value of an individual who starts a period in preference state s holding a quality j asset; a function $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ representing the buyer-seller ratio at an arbitrary price p; a function $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$ representing the distribution of asset qualities available at price p; and a nondecreasing function $F : \mathbb{R}_+ \mapsto [0, 1]$ with support $\mathbb{P}$ representing the share of assets available at a price less than or equal to p. The definition of partial equilibrium is analogous to Definition 1 for the i.i.d. case, except for the obvious change in the continuation value:

$$v_{l,j} = \delta_j + \max_p \big(\min\{\Theta(p), 1\}p + (1 - \min\{\Theta(p), 1\})\beta_l(\pi_{ll}v_{l,j} + \pi_{lh}v_{h,j})\big), \tag{1'}$$

$$v_{h,j} = \delta_j\lambda + \beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})), \tag{3'}$$

where

$$\lambda \equiv \max_p \left(\min\{\Theta(p)^{-1}, 1\} \frac{\beta_h \sum_{j=1}^J \gamma_j(p)(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})}{p} + \big(1 - \min\{\Theta(p)^{-1}, 1\}\big) \right). \tag{4'}$$

We omit the formal definition, which simply substitutes these expressions for their i.i.d. analogs. The characterization of partial equilibrium and proof that it exists and is unique is similarly unchanged. In equilibrium, quality j assets sell for a price p_j satisfying the buyers' indifference condition

$$p_j = \frac{\beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})}{\lambda},$$

while the condition for excluding quality $j-1$ assets from the market pins down the sale probability θ_j when $j \geq 2$

$$\theta_j\big(p_j - \beta_l(\pi_{ll}v_{l,j-1} + \pi_{lh}v_{h,j-1})\big) = \min\{\theta_{j-1}, 1\}\big(p_{j-1} - \beta_l(\pi_{ll}v_{l,j-1} + \pi_{lh}v_{h,j-1})\big).$$

These equations pin down the value functions, prices, and buyer-seller ratios given λ.

In the model with idiosyncratic shocks, we found that the value of the consumption good to a high discount factor individual, λ, always lies in the interval $[1, \beta_h/\beta_l]$. With persistent shocks, the lower bound, which ensures that high discount factor individuals are willing to buy assets, $p_j \leq \beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})$, is unchanged. However, the upper bound, which ensures that low discount factor individuals are willing to sell assets, $p_j \geq \beta_l(\pi_{ll}v_{l,j} + \pi_{lh}v_{h,j})$, is given by the larger root of

$$\beta_h(\lambda - (\lambda - 1)\pi_{hl}) = \beta_l\lambda(\lambda - (\lambda - 1)\pi_{ll}).$$

We denote this upper bound by $\bar{\lambda}$. It always exceeds 1 and $\bar{\lambda} > \beta_h/\beta_l$ if and only if shocks are persistent, $\pi_{ll} > \pi_{hl}$.

The definition of a competitive equilibrium with persistent shocks is also complicated by endogeneity of the distribution of asset holdings. In the i.i.d. case, high discount factor individuals start each period holding a fraction $\pi_h K_j$ quality j assets, but this is not true with persistent shocks. Instead, let μ_j denote the measure of quality j assets held by high discount factor individuals at the start of a period. In steady state, this satisfies

$$\mu_j = \pi_{hh} \left(\mu_j + \sigma_j \right) + \pi_{lh} \left(K_j - \mu_j - \sigma_j \right),$$

where σ_j is the measure of quality j assets purchased by high discount factor individuals each period. High discount factor individuals hold $\mu_j + \sigma_j$ quality j assets at the end of each period, while the rest are held by low discount factor individuals. Multiplying by the appropriate preference transition probabilities delivers the measure held by high discount factor individuals at the start of the following period. To solve for μ_j, we first need to compute the measure of assets sold each period, σ_j. This is the product of the measure of assets for sale times the average sale probability weighted by the fraction of assets that are of quality j at an arbitrary price p:

$$\sigma_j = \left(\sum_{j'} (K_{j'} - \mu_{j'}) \right) \int_{\mathbb{P}} \min\{\Theta(p), 1\} \gamma_j(p) dF(p).$$

Alternatively, consistency of supplies with beliefs implies

$$\frac{K_j - \mu_j}{\sum_{j'} (K_{j'} - \mu_{j'})} = \int_{\mathbb{P}} \gamma_j(p) dF(p),$$

and so we can rewrite the measure sold as

$$\sigma_j = (K_j - \mu_j) \frac{\int_{\mathbb{P}} \min\{\Theta(p), 1\} \gamma_j(p) dF(p)}{\int_{\mathbb{P}} \gamma_j(p) dF(p)},$$

the product of the measure of assets for sale and the average sale probability. Use this to solve for μ_j:

$$\mu_j = \frac{\pi_{lh} + (\pi_{hh} - \pi_{lh}) \frac{\int_{\mathbb{P}} \min\{\Theta(p),1\} \gamma_j(p) dF(p)}{\int_{\mathbb{P}} \gamma_j(p) dF(p)}}{1 - (\pi_{hh} - \pi_{lh}) \left(1 - \frac{\int_{\mathbb{P}} \min\{\Theta(p),1\} \gamma_j(p) dF(p)}{\int_{\mathbb{P}} \gamma_j(p) dF(p)} \right)} K_j. \tag{18}$$

If $\pi_{hh} = \pi_{lh}$, this reduces to $\mu_j = \pi_{lh} K_j = \pi_{hh} K_j$, but if shocks are persistent, $\pi_{hh} > \pi_{lh}$, then μ_j is increasing in the measure of quality j assets that are sold each period.

We are now in a position to define equilibrium:

Definition 5 *A stationary competitive equilibrium with persistent shocks is a number* $\lambda \in$

$[1, \bar{\lambda}]$, a pair of vectors $\{v_{h,j}\} \in \mathbb{R}_+^J$ and $\{v_{l,j}\} \in \mathbb{R}_+^J$, functions $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ and $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$, a nondecreasing function $F : \mathbb{R}_+ \mapsto [0,1]$ with support $\mathbb{P}$, and measures $\mu_j \in [0, K_j]$ satisfying the following conditions:

1. *$(\{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, F)$ is a partial equilibrium with persistent shocks for fixed λ;*
2. *the consumption good market clears: $\sum_{j=1}^{J} \delta_j \mu_j = \left(\sum_{j=1}^{J} (K_j - \mu_j) \right) \int_{\mathbb{P}} \Theta(p) p dF(p)$; and*
3. *measures are consistent with trades: μ_j satisfies equation (18).*

If there are a continuum of asset qualities, we can again obtain closed-form solutions. In particular, arguments analogous to those in Section 5.1 imply the price of a quality δ asset satisfies

$$P(\delta) = \frac{\beta_h \delta (\lambda \pi_{hh} + \pi_{hl} + \lambda \beta_l (\pi_{hl} - \pi_{ll}))(1 - \Theta(P(\delta)))}{\lambda(1 - \beta_l \pi_{ll} - \beta_h \pi_{hh}) - (\beta_h \pi_{hl} - \lambda \beta_l \pi_{ll}) \Theta(P(\delta)) - \lambda \beta_l \beta_h (\pi_{hl} - \pi_{ll})(1 - \Theta(P(\delta)))} \tag{12'}$$

and the sale probability satisfies

$$\Theta(p) = \frac{\lambda(1 - \beta_l(\pi_{ll} - \pi_{hl})) - (\lambda - 1)\pi_{hl}}{(\lambda - (\lambda - 1)\pi_{hl})(p/\underline{p})^{\frac{\beta_h(\lambda(1-\beta_l(\pi_{ll}-\pi_{hl}))-(\lambda-1)\pi_{hl})}{\beta_h(\lambda-(\lambda-1)\pi_{hl})-\beta_l\lambda(\lambda-(\lambda-1)\pi_{ll})}} - \beta_l \lambda (\pi_{ll} - \pi_{hl})}. \tag{13'}$$

These expressions generalize equations (12) and (13) to the model with persistent shocks. Finally, the share of assets that are of quality δ or less and are held by high discount factor individuals is

$$G_h(\delta) = \int_{\underline{\delta}}^{\delta} \frac{\pi_{lh} + (\pi_{hh} - \pi_{lh})\Theta(P(\delta'))}{1 - (\pi_{hh} - \pi_{lh})(1 - \Theta(P(\delta')))} dG(\delta'). \tag{18'}$$

We do not prove existence and uniqueness of equilibrium in this environment. For starters, extending the proof of Proposition 3 is cumbersome because the measures μ_j are endogenous and depend on λ. But this can easily be handled using the closed-form solutions when there are a continuum of asset qualities. More importantly, such a proof would only establish existence and uniqueness of a stationary competitive equilibrium, not that there is a unique equilibrium for arbitrary initial conditions. The distinction is important because μ_j is a payoff-relevant state variable in the model with persistent shocks. Given an initial value of the vector $\{\mu_j\}$, subsequent trades determine the evolution of this vector, which in turn determines the evolution of the value of consumption good to a buyer λ. We have not characterized a partial equilibrium with time-varying λ, indeed we have not even introduced notation that would allow us to do so. Therefore we cannot discuss the full set of potentially nonstationary equilibria in this environment. Nevertheless, we believe that our analysis of stationary equilibria is an important first step.

7.2 Continuous Time Limit

We are now in a position to consider the continuous time limit of this model. For a fixed period length $\Delta > 0$, define discount rates ρ_s and transition rates q_{hl} and q_{lh} as

$$\rho_s = \frac{1-\beta_s}{\Delta}, \;\; q_{hl} = \frac{\pi_{hl}}{\Delta}, \text{ and } q_{lh} = \frac{\pi_{lh}}{\Delta}.$$

Also assume a quality δ asset produces $\delta\Delta$ units of the consumption good per period. We interpret $1/\Delta$ as the number of periods within a unit of calendar time. With fixed values of ρ_s, q_{hl}, and q_{lh}, the limit as $\Delta \to 0$ (and so $\beta_s \to 1$ and π_{hl} and $\pi_{lh} \to 0$) then corresponds to the continuous time limit of the model. We find that in this limit, $\Theta(p) \to 0$ but the sale rate per unit of time converges to a number:

$$\alpha(p) \equiv \lim_{\Delta\to 0} \frac{\Theta(p)}{\Delta} = \frac{\rho_l + q_{lh} + q_{hl}/\lambda}{\left(p/\underline{p}\right)^{\frac{\rho_l + q_{lh} + q_{hl}/\lambda}{\rho_l - \rho_h - (\lambda-1)(q_{lh}+q_{hl}/\lambda)}} - 1}$$

for all $p \geq \underline{p}$, while the price of a quality δ asset converges to

$$P(\delta) = \left(\frac{\delta\big(q_{hl} + \lambda(q_{lh} + \rho_l + \alpha(P(\delta)))\big)}{\lambda\rho_h(q_{lh} + \rho_l + \alpha(P(\delta))) + q_{hl}((\lambda-1)\alpha(P(\delta)) + \lambda\rho_l)}\right).$$

In particular, the worst quality asset has dividend per unit of calendar time $\underline{\delta}$, price $P(\underline{\delta}) \equiv \underline{p}$ and no resale risk, $\alpha(\underline{p}) = \infty$. This pins down the lowest price,

$$\underline{p} = \frac{\underline{\delta}\lambda}{(\lambda-1)q_{hl} + \lambda\rho_h}.$$

From the perspective of a seller, $\alpha(p)$ is the arrival rate of a Poisson process that permits her to sell at a price p. Equivalently, the probability that she fails to sell at a price $p > \underline{p}$ during a unit of elapsed time is $\exp(-\alpha(p))$, an increasing function of p that converges to 1 as p converges to infinity and is well-behaved in the limiting economy. One can also find the arrival rate of trading opportunities to a buyer; this is infinite if $p > \underline{p}$ and zero if $p < \underline{p}$.

To close the model, we can compute the measure of quality δ assets held by high discount factor individuals, the limit of equation (18′). This gives

$$G_h(\delta) = \int_{\underline{\delta}}^{\delta} \frac{q_{lh} + \alpha(P(\delta'))}{q_{hl} + q_{lh} + \alpha(P(\delta'))} dG(\delta').$$

Substituting this into the consumption good market clearing condition gives

$$\int_{\underline{\delta}}^{\bar{\delta}} \frac{\delta(q_{lh}+\alpha(P(\delta)))}{q_{hl}+q_{lh}+\alpha(P(\delta))} dG(\delta) \geq \int_{\underline{\delta}}^{\bar{\delta}} \frac{\alpha(P(\delta))P(\delta)q_{hl}}{q_{hl}+q_{lh}+\alpha(P(\delta))} dG(\delta),$$

with equality if $\lambda > 1$. The left hand side is the integral of the dividend per unit of time δ times the density $dG_h(\delta)$, i.e. the amount of the consumption good held by high discount factor individuals at the start of a period. The integrand on the right hand side is the product of the probability per unit of time of selling a quality δ asset, $\alpha(P(\delta))$, times the price of the asset, $P(\delta)$, times the density of such assets held by low discount factor individuals, $dG(\delta) - dG_h(\delta)$. Integrating over the support of the dividend distribution gives the amount of the consumption good required to purchase the assets that are sold at each instant.

In equilibrium, there is a continuum of marketplaces, each distinguished by its price p. Sellers try to sell their assets in the appropriate market, while buyers bring their consumption good to markets and possibly consume some of it. In all but the worst market, with price $\underline{p}$, there is always too little of the consumption good to purchase all of the assets. That is, a stock of assets always remains in the market to be purchased by the gradual inflow of new buyers. Buyers are able to purchase assets immediately, but sellers and get rid of their assets only at a Poisson rate and are rationed in the sense that they wish they could sell assets faster at that price. Of course, a seller could immediately sell her assets for the low price $\underline{p}$, but she chooses not to do so.

More generally, the illiquidity generated by adverse selection does not disappear when the period length is short. Intuitively, it must take a real amount of calendar time to sell an asset at a high price or the owners of low quality assets would misrepresent them as being of high quality. This is in contrast to models where trading is slow because of search frictions.[15] In such a framework, the extent of search frictions governs the speed of trading and as the number of trading opportunities per unit of calendar time increases, the relevant frictions naturally disappear.

8 Conclusion

We have developed a dynamic model of asset trading in the presence of adverse selection. There always exists a unique separating equilibrium in which better assets sell for a higher price but in a less liquid market. We find conditions under which one type of asset is has lower liquidity, volume, and average asking price and show that this is closely linked to the

[15]See, for example, Duffie, Gârleanu and Pedersen (2005), Weill (2008), and Lagos and Rocheteau (2009) for models where assets are illiquid because of search frictions.

lower bound on the support of the quality distribution. We also show how a change in seller's private information for one asset can trigger a fire sale for that asset and a flight to other types of assets. Finally, we examined how two realistic policy proposals, an asset purchase program and an asset subsidy program, can raise an asset's price and liquidity, even if that asset is neither purchased nor subsidized by the government.

In concluding, we note that we have assumed throughout our analysis that individuals' discount factors are observable. It seems natural to ask what would happen if both asset quality and trading motives were private information. In this case, patient individuals might have an incentive to sell their low quality assets at a high price. We can prove that if $\lambda p_1 \geq p_J$ in our equilibrium, then the equilibrium allocation is unaffected by this additional source of private information.[16] Intuitively, an unobservable discount factor gives a patient individual an opportunity to buy a bad asset for p_1 and attempt to resell it for $p_J > p_1$. The reason that this trade might not be profitable is that the individual must use beginning-of-period dividends, which are worth λ to him, to purchase the asset and he only gets back the consumption good at the end of the period, which must be consumed and so is worth 1. If $\lambda p_1 < p_J$ in our equilibrium, then unobservable discount factors must change the equilibrium allocation. In our current research, we show that this can introduce a continuum of semi-pooling equilibria. In any equilibrium, individuals with different discount factors sell different types of assets at a common price (Guerrieri and Shimer, 2013).[17]

[16] It is straightforward to extend our definition of equilibrium to this environment. The only change in equilibrium involves beliefs about buyer-seller ratios at very high prices: for $p > \lambda p_1$, $\Theta(p) = 0$ and $\Gamma(p)$ is arbitrary. This implies that there is no price at which a patient individual can and would sell any of his assets.

[17] The existence of a semi-pooling equilibrium is related to Chang (2011), which develops a version of our model with two sources of private information. Sellers know both the quality of the asset they are selling and their cost of holding the asset. However, she assumes that sellers' holding costs always exceed buyers' so there are gains from trade, that an individual's identity as a buyer or seller is known, and that buyers have excess consumption good ($\lambda = 1$). We are interested in the case where there may be no gains from trade since an individual's preferences are unknown, yet scarcity of the consumption good ($\lambda > 1$) can sustain some trade.

References

Akerlof, George A., "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," *Quarterly Journal of Economics*, 1970, *84* (3), 488–500.

Arora, Sanjeev, Boaz Barak, Markus Brunnermeier, and Rong Ge, "Computational Complexity and Information Asymmetry in Financial Products," *Communications of the ACM*, 2011, *54* (5), 101–107.

Caballero, Ricardo J. and Arvind Krishnamurthy, "Collective Risk Management in a Flight to Quality Episode," *Journal of Finance*, 2008, *63* (5), 2195–2230.

Camargo, Braz and Benjamin Lester, "Trading Dynamics in Decentralized Markets with Adverse Selection," 2012. São Paulo School of Economics Mimeo.

Chang, Briana, "Adverse Selection and Liquidity Distortion in Decentralized Markets," 2011. Northwestern Mimeo.

Chari, V.V., Ali Shourideh, and Ariel Zetlin-Jones, "Adverse Selection, Reputation and Sudden Collapses in Secondary Loan Markets," 2010. University of Minnesota Mimeo.

Chiu, Jonathan and Thorsten V. Koeppl, "Trading Dynamics with Adverse Selection and Search: Market Freeze, Intervention and Recovery," 2011. Bank of Canada Mimeo.

Daley, Brendan and Brett Green, "Waiting for News in the Dynamic Market for Lemons," *Econometrica*, 2012, *80* (4), 1433–1504.

Dang, Tri Vi, Gary Gorton, and Bengt Holmström, "Ignorance and the Optimality of Debt for the Provision of Liquidity," 2009. Yale Mimeo.

DeMarzo, Peter and Darrell Duffie, "A Liquidity-Based Model of Security Design," *Econometrica*, 1999, *67* (1), 65–99.

Downing, Chris, Dwight Jaffee, and Nancy Wallace, "Is the Market for Mortgage-Backed Securities a Market for Lemons?," *Review of Financial Studies*, 2009, *22* (7), 2457–2494.

Duffie, Darrell, Nicolae Gârleanu, and Lasse H. Pedersen, "Over-the-Counter Markets," *Econometrica*, 2005, *73* (6), 1815–1847.

Easley, David and Maureen OHara, "Liquidity and Valuation in an Uncertain World," *Journal of Financial Economics*, 2010, *97* (1), 1–11.

Eisfeldt, Andrea L., "Endogenous Liquidity in Asset Markets," *Journal of Finance*, 2004, *59* (1), 1–30.

Finkelstein, Amy and James M. Poterba, "Adverse Selection and the Choice of Risk Factors in Insurance Pricing: Evidence from the UK Annuity Market," *Journal of Political Economy*, 2004, *112* (1), 183–208.

Gorton, Gary, "The Panic of 2007," *Federal Reserve Bank of Kansas City Economic Symposium Conference Proceedings*, 2008, pp. 131–262.

Guerrieri, Veronica and Robert Shimer, "Markets with Multidimensional Private Information," 2013. University of Chicago Mimeo.

_ , _ , and Randall Wright, "Adverse Selection in Competitive Search Equilibrium," *Econometrica*, 2010, *78* (6), 1823–1862.

Inderst, Roman, "Matching Markets with Adverse Selection," *Journal of Economic Theory*, 2005, *121* (2), 145–166.

Kurlat, Pablo, "Lemons, Market Shutdowns and Learning," *American Economic Review*, Forthcoming.

Lagos, Ricardo and Guillaume Rocheteau, "Liquidity in Asset Markets with Search Frictions," *Econometrica*, 2009, *77* (2), 403–426.

Milgrom, Paul and Nancy Stokey, "Information, Trade and Common Knowledge," *Journal of Economic Theory*, 1982, *26* (1), 17–27.

Rigotti, Luca and Chris Shannon, "Uncertainty and Risk in Financial Markets," *Econometrica*, 2005, *73* (1), 203–243.

Routledge, Bryan R. and Stanley E. Zin, "Model Uncertainty and Liquidity," *Review of Economic Dynamics*, 2009, *12* (4), 543–566.

Tirole, Jean, "Overcoming Adverse Selection: How Public Intervention Can Restore Market Functioning," *American Economic Review*, 2012, *102* (1), 29–59.

Weill, Pierre Olivier, "Liquidity Premia in Dynamic Bargaining Markets," *Journal of Economic Theory*, 2008, *140* (1), 66–96.

Werning, Ivan, "Principle of Optimality," 2009. MIT Mimeo.

Appendix

Omitted Proofs

Proof of Lemma 1. Consider problem (P_1). Given that there is no $j' < 1$, the only constraint is (5). If such a constraint were slack, we could increase p and hence raise the value of the objective function, which ensures the constraint binds. Eliminating the price by substituting the binding constraint into the objective function gives

$$v_{l,1} = \delta_1 + \max_{\theta} \left(\min\{\theta, 1\} \frac{\beta_h \min\{\theta^{-1}, 1\}}{\lambda - 1 + \min\{\theta^{-1}, 1\}} + (1 - \min\{\theta, 1\})\beta_l \right) \bar{v}_1.$$

If $\lambda = 1$, any $\theta_1 \geq 1$ attains the maximum. If $\lambda = \beta_h/\beta_l$, any $\theta_1 \in [0, 1]$ attains the maximum. For intermediate values of λ, the unique maximizer is $\theta_1 = 1$. Substituting back into the original problem gives $v_{l,1} = \delta_1 + p_1$ and $p_1 = \beta_h \bar{v}_1/\lambda$, establishing the result for $j = 1$.

For $j \geq 2$ we proceed by induction. Assume for all $j' \in \{2, \ldots, j-1\}$, we have established the characterization of $p_{j'}$, $\theta_{j'}$, $v_{l,j'}$ and $\bar{v}_{j'}$ in the statement of the lemma. We first prove that $\bar{v}_j > \bar{v}_{j-1}$. To do this, consider the policy (θ_{j-1}, p_{j-1}). If this solved problem (P_j), combining the objective function and the definition of $\bar{v}_j$ gives

$$\begin{aligned} \bar{v}_j &= \frac{\delta_j(\pi_h\lambda + \pi_l) + \pi_l \min\{\theta_{j-1}, 1\} p_{j-1}}{1 - \pi_h\beta_h - \pi_l\beta_l(1 - \min\{\theta_{j-1}, 1\})} \\ &> \frac{\delta_{j-1}(\pi_h\lambda + \pi_l) + \pi_l \min\{\theta_{j-1}, 1\} p_{j-1}}{1 - \pi_h\beta_h - \pi_l\beta_l(1 - \min\{\theta_{j-1}, 1\})} = \bar{v}_{j-1}. \end{aligned}$$

The inequality uses the fact that the denominator is positive together with $\delta_j > \delta_{j-1}$; and the last equality comes from the objective function and the definition of $\bar{v}_{j-1}$ in problem (P_{j-1}). Since the proposed policy satisfies all of the constraints in problem (P_{j-1}) and $\bar{v}_j > \bar{v}_{j-1}$, it also satisfies all the constraints in problem (P_j). The optimal policy must deliver a weakly higher value, proving $\bar{v}_j > \bar{v}_{j-1}$.

Next we prove that at any solution to problem (P_j) the constraint (5) is binding. If there were an optimal policy (θ, p) such that it was slack, consider a small increase in p to $p' > p$ and a reduction in θ to $\theta' < \theta$ so that $\min\{\theta, 1\}(p - \beta_l\bar{v}_{j-1}) = \min\{\theta', 1\}(p' - \beta_l\bar{v}_{j-1})$ while constraint (5) is still satisfied. Now suppose for some $j' \neq j - 1$, $\min\{\theta, 1\}(p - \beta_l\bar{v}_{j'}) < \min\{\theta', 1\}(p' - \beta_l\bar{v}_{j'})$. Subtracting the inequality from the preceding equation gives

$$\big(\min\{\theta, 1\} - \min\{\theta', 1\}\big)(\bar{v}_{j'} - \bar{v}_{j-1}) > 0.$$

Given that $\theta' < \theta$, the above inequality yields $\bar{v}_{j'} > \bar{v}_{j-1}$ and hence $j' \geq j$. This implies that the change in policy does not tighten the constraints (6) for $j' < j$, while it raises the value of the objective function in problem (P_j), a contradiction. Therefore constraint (5) must bind at the optimum.

We now show that the binding constraint (5) implies that $\theta_j \leq 1$ for all $j \geq 2$. By contradiction, assume that the solution to problem (P_j) is some (θ, p) with $\theta > 1$. In this case, the objective function reduces to $v_{l,j} = \delta_j + p$, while the constraint (6) for $j' = 1$ imposes $v_{l,1} \geq \delta_1 + p$. Since we have shown that $v_{l,1} = \delta_1 + p_1$, this implies $p \leq p_1$. Moreover, $\bar{v}_j > \bar{v}_1$ implies $\beta_h \bar{v}_j / \lambda > \beta_h \bar{v}_1 / \lambda = p_1$ and hence $\beta_h \bar{v}_j / p > \lambda$. Now a change to the policy $(1, p)$ relaxes the constraint (5) without affecting any other piece of the problem (P_j) and is therefore weakly optimal. But this cannot be optimal because (5) is slack, a contradiction. This proves that $\theta_j \leq 1$ for all $j \geq 2$ and hence, using the binding constraint (5), $p_j = \beta_h \bar{v}_j / \lambda$.

Next, we prove that if $\lambda < \beta_h / \beta_l$, the constraint (6) is binding at $j' = j - 1$. We break our proof into two parts. First, consider $j = 2$ and, to find a contradiction, assume that there is a solution (θ, p) to problem (P_2) such that constraint (6) is slack for $j' = 1$. Then problem (P_2) is equivalent to problem (P_1) except for the value of the dividend $\delta_2 > \delta_1$. Following the same argument used for problem (P_1), we can show that $\theta_2 \geq 1$ and so constraint (6) reduces to $v_{l,1} \geq \delta_1 + p_2$. But since $p_1 = \beta_h \bar{v}_1 / \lambda < p_2 = \beta_h \bar{v}_2 / \lambda$, this contradicts $v_{l,1} = \delta_1 + p_1$. Constraint (6) must bind when $j = 2$.

Next consider $j > 2$ and again assume by contradiction that there is a solution (θ, p) to problem (P_j) such that constraint (6) is slack for $j' = j - 1$. Then problem (P_j) is equivalent to problem (P_{j-1}) except in the value of the dividend δ. Since constraint (6) is binding in the solution to problem (P_{j-1}) and $\theta_{j-1} \leq 1$, we have

$$v_{l,j-2} = \delta_{j-2} + \theta_{j-1} p_{j-1} + (1 - \theta_{j-1}) \beta_l \bar{v}_{j-2} = \delta_{j-2} + \theta p + (1 - \theta) \beta_l \bar{v}_{j-2},$$

and hence

$$\theta_{j-1}(p_{j-1} - \beta_l \bar{v}_{j-2}) = \theta(p - \beta_l \bar{v}_{j-2}). \tag{19}$$

Since $p = \beta_h \bar{v}_j / \lambda$ and $p_{j-1} = \beta_h \bar{v}_{j-1} / \lambda$, $p - \beta_l \bar{v}_{j-2} > p_{j-1} - \beta_l \bar{v}_{j-2} > 0$ and so $\theta_{j-1} > \theta > 0$. But now combine equation (19) with $\theta_{j-1} > \theta$ and $\bar{v}_{j-1} > \bar{v}_{j-2}$ to get

$$\theta_{j-1}(p_{j-1} - \beta_l \bar{v}_{j-1}) < \theta(p - \beta_l \bar{v}_{j-1}).$$

This implies that constraint (6) for $j' = j - 1$ is violated, a contradiction. This proves that constraint (6) must bind whenever $\lambda < \beta_j / \beta_l$ and establishes all the equations in the statement of the lemma.

Alternatively, suppose $\lambda = \beta_h/\beta_l$. Since $p_j = \beta_h \bar{v}_j/\lambda = \beta_l \bar{v}_j$, the objective function in problem (P_j) reduces to $v_{l,j} = \delta_j + \beta_l \bar{v}_j$, while constraint (6) imposes

$$v_{l,j'} = \delta_{j'} + \beta_l \bar{v}_{j'} \geq \delta_{j'} + \beta_l \big(\theta \bar{v}_j + (1-\theta)\bar{v}_{j'}\big)$$

for all $j' < j$. Since $\bar{v}_j > \bar{v}_{j'}$, this implies $\theta = 0$ in the solution to the problem. It is easy to verify that this is implied by the equations in the statement of the lemma.

Finally, we need to prove that there is a unique value of $\bar{v}_j > \bar{v}_{j-1}$ that solves the four equations in the statement of the lemma. Combining them we obtain

$$(1 - \pi_h\beta_h - \pi_l\beta_l)\bar{v}_j = \delta_j(\pi_l + \lambda\pi_h) + \pi_l \min\{\theta_{j-1}, 1\} \frac{\big(\beta_h - \beta_l\lambda\big)^2 \bar{v}_{j-1}\bar{v}_j}{\big(\beta_h\bar{v}_j - \beta_l\lambda\bar{v}_{j-1}\big)\lambda}. \tag{20}$$

If $\lambda = \beta_h/\beta_l$, the last term is zero and so this pins down $\bar{v}_j$ uniquely. Otherwise we prove that there is a unique solution to equation (20) with $\bar{v}_j > \bar{v}_{j-1}$. In particular, the left hand side is a linearly increasing function of $\bar{v}_j$, while the right hand side is an increasing, concave function, and so there are at most two solutions to the equation. As $\bar{v}_j \to \infty$, the left hand side exceeds the right hand side, and so we simply need to prove that as $\bar{v}_j \to \bar{v}_{j-1}$, the right hand side exceeds the left hand side.

First assume $j = 2$ so $\theta_{j-1} = \theta_1 \geq 1$. Then we seek to prove that

$$(1 - \pi_h\beta_h - \pi_l\beta_l)\bar{v}_1 < \delta_2(\pi_l + \lambda\pi_h) + \pi_l \frac{\big(\beta_h - \beta_l\lambda\big)\bar{v}_1}{\lambda}.$$

Since $\bar{v}_1 = (\delta_1\lambda(\pi_l + \lambda\pi_h))/(\lambda - \beta_h(\pi_l + \lambda\pi_h))$ and $\delta_1 < \delta_2$, we can confirm this directly. Next take $j \geq 3$. In this case, in the limit with $\bar{v}_j \to \bar{v}_{j-1}$, the right hand side of (20) converges to

$$\delta_j(\pi_l + \lambda\pi_h) + \pi_l\theta_{j-1}\frac{\big(\beta_h - \beta_l\lambda\big)\bar{v}_{j-1}}{\lambda} > \delta_{j-1}(\pi_l + \lambda\pi_h) + \pi_l \min\{\theta_{j-2}, 1\} \frac{\big(\beta_h - \beta_l\lambda\big)^2 \bar{v}_{j-2}\bar{v}_{j-1}}{\big(\beta_h\bar{v}_{j-1} - \beta_l\lambda\bar{v}_{j-2}\big)\lambda},$$

where the inequality uses the indifference condition

$$\min\{\theta_{j-2}, 1\}(p_{j-2} - \beta_l\bar{v}_{j-2}) = \theta_{j-1}(p_{j-1} - \beta_l\bar{v}_{j-2})$$

and the assumption $\delta_{j-1} < \delta_j$. The right hand side of the inequality is the same as the right hand side of equation (20) for quality $j-1$. The desired inequality then follows by comparing the left hand side of the inequality to the left hand side of equation (20) for quality $j-1$.

This completes the proof. ■

Proof of Proposition 2.

We first prove that the solution to problem (P) describes a partial equilibrium and then prove that there is no other equilibrium.

Existence. As in the statement of the proposition, we look for a partial equilibrium where $\mathbb{P} = \{p_j\}$, $\Theta(p_j) = \theta_j$, $\gamma_j(p_j) = 1$, $dF(p_j) = K_j / \sum_{j'} K_{j'}$, and $v_{s,j}$ solves problem (P_j). Also for notational convenience define $p_{J+1} = \infty$. To complete the characterization, we define Θ and Γ on their full support $\mathbb{R}_+$. For $p < p_1$, $\Theta(p) = \infty$ and $\Gamma(p)$ can be chosen arbitrarily, for example $\gamma_1(p) = 1$. For $j \in \{1, \ldots, J\}$ and $p \in (p_j, p_{j+1})$, $\gamma_j(p) = 1$ and $\Theta(p)$ satisfies sellers' indifference condition $v_{l,j} = \delta_j + \min\{\Theta(p), 1\}p + \big(1 - \min\{\Theta(p), 1\}\big)\beta_l \bar{v}_j$; equivalently, $\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) = \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$. To prove that this is a partial equilibrium, we need to verify that the five equilibrium conditions hold.

To show that the third and fourth equilibrium conditions—Buyers' Optimality and Active Markets—are satisfied, it is enough to prove that the prices $\{p_j\}$ solve the optimization problem in equation (4). Lemma 1 implies that $p_j = \beta_h \bar{v}_j / \lambda$ for all λ and j; and $\Theta(p_j) \leq 1$ if $\lambda > 1$. Together these conditions imply that any price p_j achieves the maximum in this optimization problem. For any price $p \in (p_j, p_{j-1})$, $\gamma_j(p) = 1$ by construction, and so the right hand side of equation (4) is smaller than when evaluate at p_j. Moreover, for any $p < p_1$, $\Theta(p) = \infty$ and so the right hand side is $1 \leq \lambda$.

Next we prove that $\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$ for all j and p, with equality if $p \in [p_j, p_{j+1})$. The first and second equilibrium conditions—Sellers's Optimality and Equilibrium Beliefs— follow immediately from this. The equality holds by construction. Let us now focus on the inequalities.

First take any $j' \in \{2, \ldots, J\}$, $j < j'$, and $p \in [p_{j'}, p_{j'+1})$. By the construction of Θ,

$$\min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l \bar{v}_{j'}) = \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_{j'}).$$

Then $p_{j'} \leq p$ implies that $\min\{\Theta(p_{j'}), 1\} \geq \min\{\Theta(p), 1\}$. Since $j < j'$, Lemma 1 implies that $\bar{v}_{j'} > \bar{v}_j$ and so $\min\{\Theta(p_{j'}), 1\}(\bar{v}_{j'} - \bar{v}_j) \geq \min\{\Theta(p), 1\}(\bar{v}_{j'} - \bar{v}_j)$. Adding this to the previous equation gives $\min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l \bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$. Also condition (6) in problem $(P_{j'})$ implies $\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) \geq \min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l \bar{v}_j)$. Combining the last two inequalities gives $\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$ for all $p \in [p_{j'}, p_{j'+1})$ and $j < j'$.

Similarly, take any $j' \in \{1, \ldots, J-1\}$, $j > j'$, and $p \in [p_{j'}, p_{j'+1})$. The construction of Θ

implies $\min\{\Theta(p_{j'}),1\}(p_{j'}-\beta_l\bar{v}_{j'}) = \min\{\Theta(p),1\}(p-\beta_l\bar{v}_{j'})$, while Lemma 1 together with $\Theta(p_j)=\theta_j$ implies $\min\{\Theta(p_{j'}),1\}(p_{j'}-\beta_l\bar{v}_{j'}) = \min\{\Theta(p_{j'+1}),1\}(p_{j'+1}-\beta_l\bar{v}_{j'})$. The two equalities together imply

$$\min\{\Theta(p_{j'+1}),1\}(p_{j'+1}-\beta_l\bar{v}_{j'}) = \min\{\Theta(p),1\}(p-\beta_l\bar{v}_{j'})$$

Then $p_{j'+1} > p$ implies $\min\{\Theta(p_{j'+1}),1\} \leq \min\{\Theta(p),1\}$. Since $j > j'$, Lemma 1 implies that $\bar{v}_j > \bar{v}_{j'}$ and so $\min\{\Theta(p_{j'+1}),1\}(\bar{v}_{j'}-\bar{v}_j) \geq \min\{\Theta(p),1\}(\bar{v}_{j'}-\bar{v}_j)$. Adding this to the previous equation gives $\min\{\Theta(p_{j'+1}),1\}(p_{j'+1}-\beta_l\bar{v}_j) \geq \min\{\Theta(p),1\}(p-\beta_l\bar{v}_j)$. Also, since $(\Theta(p_{j'+1}),p_{j'+1})$ is a feasible policy in problem (P_j), $\min\{\Theta(p_j),1\}\ (p_j-\beta_l\bar{v}_j) \geq \min\{\Theta(p_{j'+1}),1\}(p_{j'+1}-\beta_l\bar{v}_j)$. Combining inequalities gives $\min\{\Theta(p_j),1\}(p_j-\beta_l\bar{v}_j) \geq \min\{\Theta(p),1\}(p-\beta_l\bar{v}_j)$ for all $p \in [p_{j'},p_{j'+1})$ and $j > j'$.

Finally, consider $p < p_1$. Since $\Theta(p)=\infty$, $\min\{\Theta(p),1\}(p-\beta_l\bar{v}_j) = p-\beta_l\bar{v}_j < p_1-\beta_l\bar{v}_j \leq \min\{\Theta(p_1),1\}(p_1-\beta_l\bar{v}_j)$, where the first inequality uses $p < p_1$ and the second uses the fact that $\Theta(p_1) < 1$ only if $\lambda = \beta_h/\beta_l$; but in this case, $p_1 = \beta_l\bar{v}_1 \leq \beta_l\bar{v}_j$. Since we have already proved that $\min\{\Theta(p_1),1\}(p_1-\beta_l\bar{v}_j) \leq \min\{\Theta(p_j),1\}(p_j-\beta_l\bar{v}_j)$, this establishes the inequality for $p < p_1$.

The last piece of the definition of equilibrium is Consistency of Supplies with Beliefs. This holds by the construction of the distribution function F in the statement of the Proposition.

Uniqueness. Now take any partial equilibrium $\{\{v_{h,j}\},\{v_{l,j}\},\Theta,\Gamma,F\}$. We first claim that $\bar{v}$ is increasing in j. Take $j > j'$ and let $p_{j'}$ denote the price offered by j'. Type j Sellers' Optimality implies

$$v_{l,j} \geq \delta_j + \min\{\Theta(p_{j'}),1\}p_{j'} + (1-\min\{\Theta(p_{j'}),1\})\beta_l\bar{v}_j,$$

and so combining with quality j Buyers' Optimality, equation (3), and solving for $\bar{v}_j$ gives

$$\bar{v}_j \geq \frac{\delta_j(\pi_l+\pi_h\lambda)+\pi_l\min\{\Theta(p_{j'}),1\}p_{j'}}{\pi_l(1-\min\{\Theta(p_{j'}),1\})\beta_l+\pi_h\beta_h} > \frac{\delta_{j'}(\pi_l+\pi_h\lambda)+\pi_l\min\{\Theta(p_{j'}),1\}p_{j'}}{\pi_l(1-\min\{\Theta(p_{j'}),1\})\beta_l+\pi_h\beta_h} = \bar{v}_{j'},$$

where the second inequality uses $\delta_j > \delta_{j'}$ and the equality solves the same equations for $\bar{v}_{j'}$.

Consistency of Supplies with Beliefs implies that for each $j \in \{1,\ldots,J\}$, there exists a price $p_j \in \mathbb{P}$ with $\gamma_j(p_j) > 0$.

Now in the remainder of the proof, assume also that $\theta_j \equiv \Theta(p_j) > 0$. First we prove that the constraint $\lambda \leq \min\{\theta_j^{-1},1\}\beta_h\bar{v}_j/p_j + (1-\min\{\theta_j^{-1},1\})$ is satisfied. Second we prove that the constraint $v_{l,j'} \geq \delta_{j'} + \min\{\theta_j,1\}p_j + (1-\min\{\theta_j,1\})\beta_l\bar{v}_{j'})$ is satisfied for all $j' < j$. Third we prove that the pair (θ_j,p_j) delivers value $v_{l,j}$ to sellers of quality j trees. Fourth we prove that (θ_j, p_j) solves (P_j).

Step 1. To derive a contradiction, assume $\lambda > \min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_j/p_j + 1 - \min\{\theta_j^{-1}, 1\}$. Active Markets implies that the expected value of a unit of fruit to a buyer who pays p_j must equal λ and so there must be a j' with $\gamma_{j'}(p_j) > 0$ and $\lambda < \min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_{j'}/p_j + 1 - \min\{\theta_j^{-1}, 1\}$. If $\theta_j = \infty$, $\min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_{j'}/p_j + 1 - \min\{\theta_j^{-1}, 1\} = 1 \leq \lambda$, which is impossible; therefore $\theta_j < \infty$. Then Equilibrium Beliefs implies p_j is an optimal price for quality j' sellers and so for all p' and $\theta' \equiv \Theta(p')$, $\min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'}) \geq \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j'})$. Since $\theta_j > 0$, $\min\{\theta_j, 1\}(p' - \beta_l \bar{v}_{j'}) > \min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'})$ for all $p' > p_j$, and so the two inequalities imply $\min\{\theta_j, 1\} > \min\{\theta', 1\}$.

Now take any $j'' < j'$, so $\bar{v}_{j''} < \bar{v}_{j'}$. Then since $\min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'}) \geq \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j'})$, $\min\{\theta_j, 1\} > \min\{\theta', 1\}$, and $\bar{v}_{j''} < \bar{v}_{j'}$,

$$\min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j''}) > \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j''}).$$

Type j'' Sellers' Optimality condition implies $\bar{v}_{j''} \geq \delta_{j''} + \min\{\theta_j, 1\}p_j + (1 - \min\{\theta_j, 1\})\beta_l \bar{v}_{j''}$ and so the previous inequality gives $\bar{v}_{j''} > \delta_{j''} + \min\{\theta', 1\}p' + (1 - \min\{\theta', 1\})\beta_l \bar{v}_{j''}$. Rational beliefs implies $\gamma_{j''}(p') = 0$. That is, any $p' > p_j$ attracts only quality j' sellers or higher and so delivers value at least equal to $\min\{\theta'^{-1}, 1\}\beta_h \bar{v}_{j'}/p' + (1 - \min\{\theta'^{-1}, 1\})$ to buyers. For p' sufficiently close to p_j, this exceeds λ, contradicting buyers' optimality.

Step 2. Sellers' Optimality implies $v_{l,j'} \geq \delta_{j'} + \min\{\theta_j, 1\}p_j + (1 - \min\{\theta_j, 1\})\beta_l \bar{v}_{j'})$ for all j', p_j, and $\theta_j = \Theta(p_j)$.

Step 3. Equilibrium Beliefs implies $v_{l,j} = \delta_j + \min\{\theta_j, 1\}p_j + (1 - \min\{\theta_j, 1\})\beta_l \bar{v}_{j'})$ for all j, p_j, and $\theta_j = \Theta(p_j) < \infty$ with $\gamma_j(p_j) > 0$.

Step 4. Suppose there is a policy (θ, p) that satisfies the constraints of problem (P_j) and delivers a higher payoff. That is,

$$v_{l,j} < \delta_j + \min\{\theta, 1\}p + (1 - \min\{\theta, 1\})\beta_l \bar{v}_j$$
$$\lambda \leq \min\{\theta^{-1}, 1\}\beta_h \bar{v}_j/p + 1 - \min\{\theta^{-1}, 1\}$$
$$v_{l,j'} \geq \delta_{j'} + \min\{\theta, 1\}p + (1 - \min\{\theta, 1\})\beta_l \bar{v}_{j'} \text{ for all } j' < j.$$

If these inequalities hold with $\theta > 1$, then the same set of inequalities holds with $\theta = 1$, and so we may assume $\theta \leq 1$ without loss of generality. Choose $p' < p$ such that

$$v_{l,j} < \delta_j + \theta p' + (1 - \theta)\beta_l \bar{v}_j \tag{21}$$
$$\lambda < \beta_h \bar{v}_j/p' \tag{22}$$
$$v_{l,j'} > \delta_{j'} + \theta p' + (1 - \theta)\beta_l \bar{v}_{j'} \text{ for all } j' < j. \tag{23}$$

The previous inequalities imply that this is always feasible by setting p' close enough to p. Now sellers' optimality implies $v_{l,j} \geq \delta_j + \min\{\Theta(p'), 1\}p' + (1 - \min\{\Theta(p'), 1\})\beta_l \bar{v}_j)$, which, together with inequality (21), implies $\Theta(p') < \theta$. This together with inequality (23) implies that

$$v_{l,j'} > \delta_{j'} + \Theta(p')p' + (1 - \Theta(p'))\beta_l \bar{v}_{j'} \text{ for all } j' < j,$$

and so, due to Equilibrium Beliefs, $\gamma_{j'}(p') = 0$ for all $j' < j$. But then, using inequality (22), we obtain

$$\lambda < \frac{\beta_h \bar{v}_j}{p'} \leq \frac{\beta_h \sum_{j'=1}^{J} \gamma_{j'}(p')\bar{v}_{j'}}{p'} = \min\{\Theta(p')^{-1}, 1\}\frac{\beta_h \sum_{j'=1}^{J} \gamma_{j'}(p')\bar{v}_{j'}}{p'} + \left(1 - \min\{\Theta(p')^{-1}, 1\}\right),$$

where the second inequality uses monotonicity of $\bar{v}_j$ and $\gamma_{j'}(p') = 0$ for $j' < j$; and the last equation uses $\Theta(p') < \theta \leq 1$. This contradicts Buyers' Optimality condition and completes the proof. ■

Proof of Proposition 3. To prove that there exists a unique competitive equilibrium, it is enough to prove that there exists a unique $\lambda \in [1, \beta_h/\beta_l]$ such that the partial equilibrium associated to that λ clears the fruit market.

For given $\lambda \in [1, \beta_h/\beta_l]$, let $x_j(\lambda) \equiv \theta_j(\lambda)p_j(\lambda)$, where $\theta_j(\lambda)$ and $p_j(\lambda)$ are the partial equilibrium sale probability and price for quality j trees. For all $j > 1$ and given $x_{j-1}(\lambda)$, define

$$f_j(x_j, \lambda) \equiv x_j \left[1 - \frac{\beta_l}{\beta_h}\lambda \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)}\right] - x_{j-1}(\lambda)\left[1 - \frac{\beta_l}{\beta_h}\lambda\right],$$

where, with some abuse of notation,

$$p_j(x_j, \lambda) = \frac{\delta_j \beta_h(\pi_l + \lambda\pi_h) + x_j \pi_l[\beta_h - \beta_l \lambda]}{\lambda(1 - \bar{\beta})}. \tag{24}$$

Given $\lambda \in [1, \beta_h/\beta_l]$, Proposition 2 and Lemma 1 ensure that $p_j(x_j(\lambda), \lambda)$ is the equilibrium price for quality j trees with $x_j(\lambda)$ being implicitly defined by $f_j(x_j, \lambda) = 0$ for all $j > 1$. Moreover, for $\lambda \in (1, \beta_h/\beta_l)$

$$x_1(\lambda) = p_1(x_1(\lambda), \lambda) = \frac{\delta_1 \beta_h(\pi_l + \lambda\pi_h)}{\lambda - \beta_h(\pi_l + \lambda\pi_h)}. \tag{25}$$

Lemma 1 also implies that $p_j(x_j(\lambda), \lambda) > p_{j-1}(x_{j-1}(\lambda), \lambda)$ for all $j > 1$. From $f_j(x_j, \lambda) = 0$ for all $j > 1$ immediately follows that $x_j(\lambda) < x_{j-1}(\lambda)$ for all $j > 1$.

Next, define $M(\lambda)$ as

$$M(\lambda) \equiv \sum_{j=1}^{J} [\pi_h \delta_j - \pi_l x_j(\lambda)] K_j.$$

Market clearing requires $M(\lambda) = 0$. Now we show that $x_j'(\lambda) < 0$ and hence $M'(\lambda) > 0$ for all $\lambda \in (1, \beta_h/\beta_l)$. For $j = 1$ we can directly calculate

$$x_1'(\lambda) = -\frac{\delta_1 \beta_h \pi_l}{[\lambda - \beta_h(\pi_l + \lambda \pi_h)]^2} < 0.$$

For all $j > 1$, given $x_{j-1}'(\lambda) < 0$ we can proceed recursively as follows. Applying the implicit function theorem to $f_j(x_j, \lambda) = 0$, we obtain

$$x_j'(\lambda) = -\frac{\partial f_j(x_j, \lambda)/\partial \lambda}{\partial f_j(x_j, \lambda)/\partial x_j}.$$

First, we can calculate

$$\frac{\partial f_j(x_j, \lambda)}{\partial x_j} = 1 - \frac{\beta_l}{\beta_h}\lambda \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)} + x_j \frac{\beta_l}{\beta_h}\lambda \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)^2} \frac{\partial p_j(x_j, \lambda)}{\partial x_j}.$$

It is easy to show that $\partial f_j(x_j, \lambda)/\partial x_j > 0$ given that $p_j(x_j(\lambda), \lambda) > p_{j-1}(x_{j-1}(\lambda), \lambda)$ and

$$\frac{\partial p_j(x_j, \lambda)}{\partial x_j} = \frac{\pi_l[\beta_h - \beta_l \lambda]}{\lambda(1 - \bar{\beta})} > 0.$$

Second, we can calculate

$$\begin{aligned}
\frac{\partial f_j(x_j, \lambda)}{\partial \lambda} =& \frac{\beta_l}{\beta_h}\left[x_{j-1}(\lambda) - x_j \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)}\right] - \left(1 - \frac{\beta_l}{\beta_h}\lambda\right) x_{j-1}'(\lambda) \\
&- \frac{\beta_l}{\beta_h}\lambda \frac{x_j}{p_j(x_j, \lambda)} \frac{\partial p_{j-1}(x_{j-1}(\lambda), \lambda)}{\partial x_{j-1}(\lambda)} x_{j-1}'(\lambda) \\
&- \frac{\beta_l}{\beta_h}\lambda \frac{x_j}{p_j(x_j, \lambda)} \left[\frac{\partial p_{j-1}(x_{j-1}(\lambda), \lambda)}{\partial \lambda} - \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)} \frac{\partial p_j(x_j, \lambda)}{\partial \lambda}\right]
\end{aligned}$$

where the first term is positive because $x_j(\lambda) < x_{j-1}(\lambda)$ and $p_j(x_j(\lambda), \lambda) > p_{j-1}(x_{j-1}(\lambda), \lambda)$, the second term is positive because $\lambda \in (1, \beta_h/\beta_l)$ and $x_{j-1}'(\lambda) < 0$, and the third term is positive because of the last inequality together with $\partial p_j(x_j, \lambda)/\partial \lambda > 0$. Finally, to show that the last term is also positive we need to show that the term in square bracket is positive where

$$\frac{\partial p_j(x_j, \lambda)}{\partial \lambda} = -\frac{\beta_h \pi_l(\delta_j + x_j)}{\lambda^2(1 - \bar{\beta})}.$$

Using expression (24) for $p_j(x_j,\lambda)$ and $f_j(x_j,\lambda) = 0$ for all j, after some algebra, one can show that this is always the case given that $\lambda \in (1, \beta_h/\beta_l)$. This implies that $x_j'(\lambda) < 0$ for all j and hence $M'(\lambda) > 0$.

Finally, define

$$\underline{\pi} \equiv \frac{\sum_{j=1}^{J} \delta_j K_j}{\sum_{j=1}^{J}[\delta_j + x_j(0)]K_j} \text{ and } \overline{\pi} \equiv \frac{\sum_{j=1}^{J} \delta_j K_j}{\sum_{j=1}^{J}[\delta_j + x_j(\beta_h/\beta_l - 1)]K_j},$$

where $x_1(\lambda)$ is given in equation (25) and $x_j(\lambda)$ solves $f_j(x_j,\lambda) = 0$ for all $j > 1$. It is easy to see that $\underline{\pi} < \overline{\pi}$ given that $x_j'(\lambda) < 0$. Moreover, $M(0) < 0$ iff $\pi_l > \underline{\pi}$ and $M(\beta_h/\beta_l - 1) > 0$ iff $\pi_l < \overline{\pi}$. Given that $M'(\lambda) > 0$, it follows that if $\pi_l \in (\underline{\pi}, \overline{\pi})$, there exists a unique equilibrium with $\lambda \in (1, \beta_h/\beta_l)$. If instead $\pi_l \leq \underline{\pi}$, then both $M(0)$ and $M(\beta_h/\beta_l - 1)$ are larger than zero, while if $\pi_l \geq \overline{\pi}$, they are both smaller than zero. Lemma 1 implies that $x_1(\lambda) \geq p_1(\lambda)$ if $\lambda = 1$ and $x_1(\lambda) \leq p_1(\lambda)$ if $\lambda = \beta_h/\beta_l$. This implies that if $\pi_l \leq \underline{\pi}$, there exists a unique equilibrium with $\lambda = 1$, where $x_1(0) \geq p_1(0)$ is pinned down by market clearing. If instead $\pi_l \geq \overline{\pi}$, then there exists a unique equilibrium with $\lambda = \beta_h/\beta_l$, where $x_1(0) \leq p_1(0)$ is pinned down by market clearing. This completes the proof. ■

Proof of Proposition 4. It is straightforward to verify that if P_1 and Θ_1 satisfy equations (12) and (13) with $\underline{p}_1 = P_1(\underline{\delta})$, then the specified functions P_2 and Θ_2 satisfy the same pair of equations with $\underline{p}_2 = P_2(\kappa\underline{\delta})$. The remaining results follow directly from the definition of liquidity, volume, asking price, and transaction price. ■

Proof of Proposition 5. By construction, $\underline{\delta}_2 = \underline{\delta}_1\underline{\varepsilon}$. Now equation (12) implies $\underline{p}_2 = \underline{p}_1\underline{\varepsilon}$ since $\Theta_2(\underline{p}_2) = \Theta_1(\underline{p}_1) = 1$. Then equation (13) implies $\Theta_1(p) = \Theta_2(p\underline{\varepsilon})$ for all $p > 0$. Finally, equation (12) implies $\Theta_1(P_1(\delta)) = \Theta_2(P_2(\delta\underline{\varepsilon}))$ and $P_2(\delta\underline{\varepsilon}) = P_1(\delta)\underline{\varepsilon}$ for all $\delta > 0$. We use these relationships throughout our proof, which we break into pieces corresponding to the four claims.

Liquidity. The liquidity of type 2 assets is

$$L_2 = \pi_l \int_{\underline{\delta}_1}^{\bar{\delta}_1} \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta_1\varepsilon))h(\varepsilon)g_1(\delta_1)d\varepsilon d\delta_1 < \pi_l \int_{\underline{\delta}_1}^{\bar{\delta}_1} \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta_1\underline{\varepsilon}))h(\varepsilon)g_1(\delta_1)d\varepsilon d\delta_1 = L_1.$$

The inequality uses the fact that P_2 is increasing and Θ_2 is decreasing. The second equality uses $\Theta_2(P_2(\delta_1\underline{\varepsilon})) = \Theta_1(P_1(\delta_1))$ and integrates over ε.

Volume. We have proved that $L_2 < L_1$. Below we prove $T_2 < T_1$. The definition of V_a then implies $V_2 < V_1$.

Asking Price. The average asking price of type 2 assets is

$$A_2 = \int_{\underline{\delta}_1}^{\bar{\delta}_1} \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} P_2(\delta_1 \varepsilon) h(\varepsilon) g_1(\delta_1) d\varepsilon d\delta_1 < \int_{\underline{\delta}_1}^{\bar{\delta}_1} \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} P_2(\delta_1 \underline{\varepsilon}) \frac{\varepsilon}{\underline{\varepsilon}} h(\varepsilon) g_1(\delta_1) d\varepsilon d\delta_1 = A_1.$$

The inequality uses the fact that price-dividend ratio is lower for higher quality assets, that is $P_2(\delta)/\delta$ is decreasing in δ since $\Theta(p)$ is decreasing (equation 12). The equality uses $P_1(\delta_1) = P_2(\delta_1 \underline{\varepsilon})/\underline{\varepsilon}$ and integrates over ε.

Transaction Price. We break this into two pieces. Consider first the average transaction price for all type 2 assets with dividend $\delta_2 = \delta\varepsilon$, conditional on the value of δ. This is

$$T_2(\delta) \equiv \frac{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon)) P_2(\delta\varepsilon) h(\varepsilon) d\varepsilon}{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon)) h(\varepsilon) d\varepsilon}.$$

We again use the fact that the price-dividend ratio is decreasing in δ and hence $P_2(\delta\varepsilon)/\varepsilon < P_2(\delta\underline{\varepsilon})/\underline{\varepsilon} = P_1(\delta)$ for all $\varepsilon > \underline{\varepsilon}$ to get

$$T_2(\delta) < P_1(\delta) \frac{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon)) \varepsilon h(\varepsilon) d\varepsilon}{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon)) h(\varepsilon) d\varepsilon}.$$

Now, because the function $\Theta_2(P_2(\delta\varepsilon))$ is decreasing in ε, the likelihood ratio $\frac{\Theta_2(P_2(\delta\varepsilon))h(\varepsilon)}{h(\varepsilon)}$ is monotone decreasing. It follows that the generalized density $\Theta_2(P_2(\delta\varepsilon))h(\varepsilon)$ is first order stochastically dominated by the density $h(\varepsilon)$. This implies that the preceding expression is smaller than

$$P_1(\delta) \frac{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \varepsilon h(\varepsilon) d\varepsilon}{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} h(\varepsilon) d\varepsilon} = P_1(\delta),$$

since the expected value of ε is 1. This proves $T_2(\delta) < P_1(\delta)$.

Now express the average transaction price for type 2 assets as a weighted average of the average transaction price conditional on δ:

$$T_2 = \frac{\int_{\underline{\delta}}^{\bar{\delta}} \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon)) h(\varepsilon) d\varepsilon \right) T_2(\delta) g_1(\delta) d\delta}{\int_{\underline{\delta}}^{\bar{\delta}} \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon)) h(\varepsilon) d\varepsilon \right) g_1(\delta) d\delta} < \frac{\int_{\underline{\delta}}^{\bar{\delta}} \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon)) h(\varepsilon) d\varepsilon \right) P_1(\delta) g_1(\delta) d\delta}{\int_{\underline{\delta}}^{\bar{\delta}} \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon)) h(\varepsilon) d\varepsilon \right) g_1(\delta) d\delta}, \tag{26}$$

where the inequality uses $T_2(\delta) < P_1(\delta)$. Consider how the difficulty of selling a type $\delta\varepsilon$ asset changes with δ and ε. Using the functional form for Θ in equation (13),

$$\frac{\partial \log \Theta_2(P_2(\delta\varepsilon))}{\partial \delta} = -\frac{\beta_h}{\beta_h - \lambda\beta_l} \frac{\varepsilon P_2'(\delta\varepsilon)}{P_2(\delta\varepsilon)}.$$

We prove that this is decreasing in ε, i.e. that $\Theta_2(P_2(\delta\varepsilon))$ is log-submodular in δ and ε. This is true if and only if the elasticity of P_2 is increasing. Implicitly differentiate equation (12) and use the functional form for $\Theta(p)$ in equation (13) to get

$$\frac{\delta P_a'(\delta)}{P_a(\delta)} = \left[1 + \frac{\pi_l}{\pi_l + \lambda\pi_h} \frac{P_a(\delta)}{\delta} \Theta_a(P_a(\delta))\right]^{-1}$$

This elasticity is increasing in δ since both $P_a(\delta)/\delta$ and $\Theta_a(P_a(\delta))$ are decreasing. This establishes log-submodularity of $\Theta_2(P_2(\delta\varepsilon))$. Equivalently, for any $\delta_1 < \delta_2$ and $\underline{\varepsilon} < \varepsilon$,

$$\frac{\Theta_2(P_2(\delta_1\varepsilon))}{\Theta_2(P_2(\delta_1\underline{\varepsilon}))} > \frac{\Theta_2(P_2(\delta_2\varepsilon))}{\Theta_2(P_2(\delta_2\underline{\varepsilon}))}$$

Weighting by $h(\varepsilon)$ and integrating over $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$, this implies

$$\frac{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon))h(\varepsilon)d\varepsilon}{\Theta_2(P_2(\delta\underline{\varepsilon}))}$$

is decreasing in δ. Once again, since the relevant likelihood ratio is monotone, the generalized density $\left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon))h(\varepsilon)d\varepsilon\right)g_1(\delta)$ is first order stochastically dominated by the generalized density $\Theta_2(P_2(\delta\underline{\varepsilon}))g_1(\delta)$. Since $P_1(\delta)$ is increasing, this implies

$$\frac{\int_{\underline{\delta}}^{\bar{\delta}} \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon))h(\varepsilon)d\varepsilon\right) P_1(\delta)g_1(\delta)d\delta}{\int_{\underline{\delta}}^{\bar{\delta}} \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon))h(\varepsilon)d\varepsilon\right) g_1(\delta)d\delta} < \frac{\int_{\underline{\delta}}^{\bar{\delta}} \Theta_2(P_2(\delta\underline{\varepsilon}))P_1(\delta)g_1(\delta)d\delta}{\int_{\underline{\delta}}^{\bar{\delta}} \Theta_2(P_2(\delta\underline{\varepsilon}))g_1(\delta)d\delta}.$$

The left hand side is bigger than T_2 by inequality (26), while $\Theta_2(P_2(\delta\underline{\varepsilon})) = \Theta_1(P_1(\delta))$ implies that the right hand side is equal to T_1. ■

Proof of Proposition 6. Since the two assets have the same support, the pricing function $P(\delta)$ and the buyer-seller ratio $\Theta(p)$ are the same as well. Implicitly differentiate $P(\delta)$ using equation (12) and the functional form for $\Theta(p)$ in equation (13) to prove the price function is convex:

$$P''(\delta) = \frac{\pi_l\beta_l\beta_h\Theta(P(\delta))P'(\delta)^2}{(\beta_h - \lambda\beta_l)\big(1 - \pi_h\beta_h - \pi_l\beta_l(1 - \Theta(P(\delta)))\big)P(\delta)} > 0.$$

Similarly, differentiate $\Theta(P(\delta))$ and simplify using the same equations:

$$\frac{d^2\Theta(P(\delta))}{d\delta^2} = \frac{P''(\delta)}{P(\delta)\pi_l\beta_l}\left(\frac{\beta_h(1-\pi_h\beta_h-\pi_l\beta_l)}{\beta_h-\lambda\beta_l} + 1 - \pi_h\beta_h - \pi_l\beta_l(1-\Theta(P(\delta)))\right) > 0.$$

Finally, implicitly differentiate $\Theta(P(\delta))P(\delta)$:

$$\frac{d^2\Theta(P(\delta))P(\delta)}{d\delta^2} = \frac{\lambda P''(\delta)(1-\pi_h\beta_h-\pi_l\beta_l)}{\pi_l(\beta_h-\lambda\beta_l)} > 0$$

Since all three of these functions are convex and G_1 second order stochastically dominates G_2, the result immediately follows from the definition of liquidity, volume, and average asking price. ■

Proof of Lemma 2. Differentiate equations (12) and (13) to prove that for all a and δ, $P_a(\delta)$ and $\Theta_a(P_a(\delta))$ are decreasing functions of λ. We find that

$$\begin{aligned}\frac{\partial P_a(\delta)}{\partial\lambda} = {} & \frac{-\pi_l P_a(\delta)}{\beta_h(\lambda\pi_h+\pi_l)^2\left(\frac{\delta}{P_a(\delta)}+\frac{\Theta_a(P_a(\delta))\pi_l}{\lambda\pi_h+\pi_l}\right)} \\ & \times\Bigg(\frac{\Theta_a(P_a(\delta))\big(\pi_l\beta_h(1-\pi_h\beta_h-\pi_l\beta_l)+\lambda(1-\pi_h\beta_h)(\pi_h\beta_h+\pi_l\beta_l)\big)}{\lambda(1-\beta_h(\pi_h+\pi_l/\lambda))} \\ & \qquad + (1-\pi_h\beta_h-\pi_l\beta_l) - \beta_l(\lambda\pi_h+\pi_l)\Theta_a(P_a(\delta))\log\Theta_a(P_a(\delta))\Bigg) < 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\Theta_a(P_a(\delta))}{\partial\lambda} = {} & \frac{-\Theta_a(P_a(\delta))}{\lambda(\beta_h-\lambda\beta_l)\big(1-\pi_h\beta_h-\pi_l\beta_l(1-\Theta_a(P_a(\delta)))\big)} \\ & \times\left(\frac{\pi_l\beta_h^2(1-\Theta_a(P_a(\delta)))(1-\pi_h\beta_h-\pi_l\beta_l)}{\lambda(1-\beta_h(\pi_h+\pi_l/\lambda))} - \frac{\delta\beta_l\beta_h(\pi_l+\lambda\pi_h)\log\Theta_a(P_a(\delta))}{P_a(\delta)}\right) < 0.\end{aligned}$$

It follows immediately that liquidity, volume, and average asking price are decreasing in λ.

To prove average transaction price is decreasing in λ, we use the fact that when λ is higher, Θ_a falls more for higher values of δ, and so the generalized density $\Theta_a(P_a(\delta))dG_a(\delta)$ is higher in the sense of first order stochastic dominance when λ is lower. Again directly using equations (12), and (13), we obtain

$$
\begin{aligned}
\frac{\partial(\Theta_a(P_a(\delta_1))/\Theta_a(P_a(\delta_2)))}{\partial\lambda} = \frac{\beta_h}{z(\beta_h-\lambda\beta_l)}\bigg(&-\frac{\beta_l\beta_h}{(\beta_h-\lambda\beta_l)^2}\log z \\
&+\frac{\pi_l}{\lambda^2(1-\beta_h(\pi_h+\pi_l/\lambda))\big(1-\pi_h\beta_h-\pi_l\beta_l(1-\theta_1)\big)\big(1-\pi_h\beta_h-\pi_l\beta_l(1-z\theta_1)\big)} \\
&\times\Big(\beta_h\theta_1(1-\pi_h\beta_h)(1-\pi_h\beta_h-\pi_l\beta_l)(1-z+z\log z) \\
&-\theta_1\big(\pi_l\beta_l^2\lambda(1-\beta_h(\pi_h+\pi_l/\lambda))(1-\theta_1)+(\beta_h-\lambda\beta_l)(1-\pi_h\beta_h)^2\big)z\log z \\
&-\lambda\beta_l(1-\beta_h(\pi_h+\pi_l/\lambda))(1-\pi_h\beta_h-\pi_l\beta_l)(1-z)\theta_1\log\theta_1\Big)\bigg),
\end{aligned}
$$

where $\theta_1 \equiv \Theta_a(P_a(\delta_1))$ and $z \equiv \Theta_a(P_a(\delta_2))/\Theta_a(P_a(\delta_1))$. If $\delta_1 < \delta_2$, $z < 1$ and $\theta_1 \leq 1$, and so one can verify that each line in this expression is nonnegative and all but the last is positive. Thus higher values of λ both reduce the price of each quality and increase the relative likelihood that low quality goods are sold, reducing the average transaction price. ■

Proof of Proposition 8. It is straightforward to prove that the expressions in the statement of the proposition describe an equilibrium. We again can use the arguments in Guerrieri and Shimer (2013) to prove that this is the unique equilibrium.

The next step in the proof is based on comparative statics in $\hat{p}$:

$$
\frac{\partial\log\tilde{P}(\delta)}{\partial\log\hat{p}} = \frac{\pi_l\beta_h\tilde{\Theta}(\tilde{P}(\delta))}{\lambda\big(1-\pi_h\beta_h-\pi_l\beta_l(1-\tilde{\Theta}(\tilde{P}(\delta)))\big)} > 0.
$$

In addition, it is straightforward to verify that $\hat{\Theta}(\hat{P}(\gamma\hat{\delta})) = \Theta(P(\gamma\underline{\delta}))$ for all $\gamma > 1$. Since $\hat{\Theta}(\hat{P}(\delta))$ is a decreasing function, $\hat{\delta} > \underline{\delta}$ implies $\Theta(P(\gamma\underline{\delta})) < \hat{\Theta}(\hat{P}(\gamma\underline{\delta}))$ for all γ, so liquidity of all assets is higher under the asset purchase program. ■

Proof of Proposition 9. Throughout this proof, let $\hat{G}(\delta) \equiv \frac{G(\delta)-G(\hat{\delta})}{1-G(\hat{\delta})}$ denote the quality distribution of the asset in the private market, with associated density $\hat{g}$.

Average Asking Price. By definition,

$$
\hat{A} = \int_{\hat{\delta}}^{\bar{\delta}}\hat{P}(\delta)\hat{g}(\delta)d\delta > \int_{\hat{\delta}}^{\bar{\delta}}P(\delta)\hat{g}(\delta)d\delta > \int_{\underline{\delta}}^{\bar{\delta}}P(\delta)g(\delta)d\delta = A,
$$

where the first inequality uses the fact that for every quality δ, $\hat{P}(\delta) > P(\delta)$ and the second uses the fact that $\hat{G}(\delta)$ first order stochastically dominates $G(\delta)$ and $P(\delta)$ is increasing.

Average Transaction Price. Let $D(p)$ and $\hat{D}(p)$ denote the inverse of $P(\delta)$ and $\hat{P}(\delta)$ respectively. Then

$$\hat{T} = \frac{\int_{\hat{p}}^{\hat{P}(\bar{\delta})} \hat{\Theta}(p)p\, dG(\hat{D}(p))}{\int_{\hat{p}}^{\hat{P}(\bar{\delta})} \hat{\Theta}(p)\, dG(\hat{D}(p))} > \frac{\int_{\hat{p}}^{P(\bar{\delta})} \Theta(p)p\, dG(D(p))}{\int_{\hat{p}}^{P(\bar{\delta})} \Theta(p)\, dG(D(p))} > T.$$

The first inequality uses the fact that $\hat{P}(\delta) > P(\delta)$ for all $\delta > \hat{\delta}$ implies $\hat{D}(p) < D(p)$ for all $p > \hat{p}$, and so $G(\hat{D}(p)) < G(D(p))$, i.e. the first distribution first order stochastically dominates the second. Since $\hat{\Theta}(p) \propto \Theta(p)$, the generalized density $\hat{\Theta}(p)dG(\hat{D}(p))$ therefore first order stochastically dominates $\Theta(p)dG(D(p))$. The second inequality holds because T is a weighted average of the expected price conditional on $p > \hat{p}$ and a smaller number, the expected price conditional on $p \in [\underline{p}, \hat{p}]$.

Liquidity. By definition,

$$\hat{L} = \pi_l \int_{\hat{\delta}}^{\infty} \hat{\Theta}(\hat{P}(\delta))\hat{g}(\delta)d\delta = \pi_l \int_{1}^{\infty} \Theta(P(\gamma\underline{\delta}))\hat{g}(\gamma\hat{\delta})d\gamma,$$

since $\hat{\Theta}(\hat{P}(\gamma\hat{\delta})) = \Theta(P(\gamma\underline{\delta}))$. Next, the log concavity condition is equivalent to $\frac{\delta G'(\delta)}{1-G(\delta)}$ nondecreasing. This in turn holds if and only if $(G(\gamma\delta) - G(\delta))/(1 - G(\delta))$ is nondecreasing in δ for all γ. Therefore $\hat{G}(\gamma\hat{\delta}) \geq G(\gamma\underline{\delta})$ for all $\gamma > 1$, so the distribution relevant for $\hat{L}$ is first order stochastically dominated by the distribution relevant for L. Since Θ is decreasing, this implies

$$\int_{1}^{\infty} \Theta(P(\gamma\underline{\delta}))\hat{g}(\gamma\hat{\delta})d\gamma \geq \int_{1}^{\infty} \Theta(P(\gamma\underline{\delta}))g(\gamma\underline{\delta})d\gamma) = L.$$

This proves $\hat{L} > L$.

Volume. This follows immediately from the increase in liquidity and average transaction price. ■

Proof of Proposition 10.

Prices. Since $\Theta(P(\underline{\delta})) = \hat{\Theta}(\hat{P}(\underline{\delta})) = 1$, equations (12) and (16) imply $\hat{P}(\underline{\delta}) > P(\underline{\delta})$. Equation (13) implies Θ is decreasing and so $\Theta(\hat{P}(\underline{\delta})) < 1$. Next equations (13) and (17) imply that $\hat{\Theta}'(p)/\hat{\Theta}(p) > \Theta'(p)/\Theta(p)$ for all p and so $\hat{\Theta}(p) > \Theta(p)$ for all $p \geq \hat{P}(\underline{\delta})$.

Now suppose to find a contradiction that $\hat{P}(\delta) \leq P(\delta)$ for some $\delta \geq \underline{\delta}$. Then since $\hat{\Theta}(p) > \Theta(p)$ for all $p \geq \hat{P}(\underline{\delta})$ and both are decreasing functions, $\hat{\Theta}(\hat{P}(\delta)) > \Theta(P(\delta))$. Since $\sigma(\hat{P}(\delta)) \geq 0$,

$$\frac{\delta(\pi_l + \lambda\pi_h) + \pi_l\hat{\Theta}(\hat{P}(\delta))\sigma(\hat{P}(\delta))}{\lambda(1 - \pi_l\beta_l - \pi_h\beta_h) - \pi_l(\beta_h - \beta_l\lambda)\hat{\Theta}(\hat{P}(\delta))} > \frac{\delta(\pi_l + \lambda\pi_h)}{\lambda(1 - \pi_l\beta_l - \pi_h\beta_h) - \pi_l(\beta_h - \beta_l\lambda)\Theta(P(\delta))}.$$

Using equations (12) and (16), this implies $\hat{P}(\delta) > P(\delta)$, a contradiction.

Sale Probabilities. Totally differentiate equation (16) with respect to δ. Then use equation (17) to prove that

$$\hat{P}'(\delta) = \frac{\beta_h(\lambda\pi_h + \pi_l)}{\lambda\big(1 - \beta_h\pi_h - \beta_l\pi_l(1 - \hat{\Theta}(\hat{P}(\delta)))\big)},$$

a function of the subsidy only indirectly through $\hat{\Theta}(\hat{P}(\delta))$. Then using equation (17), we obtain

$$\frac{\partial \log \hat{\Theta}(\hat{P}(\delta))}{\partial \delta} = \frac{-\beta_h^2(\lambda\pi_h + \pi_l)(1 + \sigma'(\hat{P}(\delta)))}{\lambda\big(1 - \beta_h\pi_h - \beta_l\pi_l(1 - \hat{\Theta}(\hat{P}(\delta)))\big)\big(\hat{P}(\delta)(\beta_h - \lambda\beta_l) + \beta_h\sigma(\hat{P}(\delta))\big)},$$

and similarly for $\frac{\partial \log \Theta(P(\delta))}{\partial \delta}$. Since $\hat{\Theta}(\hat{P}(\underline{\delta})) = \Theta(P(\underline{\delta})) = 1$, $\hat{P}(\underline{\delta}) > P(\underline{\delta})$, and $\sigma(\hat{P}(\underline{\delta})) > 0 > \sigma'(\hat{P}(\underline{\delta}))$, this proves $\frac{\partial \log \hat{\Theta}(\hat{P}(\underline{\delta}))}{\partial \delta} > \frac{\partial \log \Theta(P(\underline{\delta}))}{\partial \delta}$. The same logic implies that at any $\delta > \underline{\delta}$, if $\hat{\Theta}(\hat{P}(\delta)) \geq \Theta(P(\delta))$, then $\frac{\partial \log \hat{\Theta}(\hat{P}(\delta))}{\partial \delta} > \frac{\partial \log \Theta(P(\delta))}{\partial \delta}$. This implies $\hat{\Theta}(\hat{P}(\delta)) > \Theta(P(\delta))$ for all $\delta > \underline{\delta}$, as we prove in the next paragraph.

First, note that $\hat{\Theta}(\hat{P}(\underline{\delta})) = \Theta(P(\underline{\delta}))$ and $\frac{\partial \log \hat{\Theta}(\hat{P}(\underline{\delta}))}{\partial \delta} > \frac{\partial \log \Theta(P(\underline{\delta}))}{\partial \delta}$ implies that there exists an $\varepsilon > 0$ such that for all $\delta \in (\underline{\delta}, \underline{\delta} + \varepsilon)$, $\hat{\Theta}(\hat{P}(\delta)) > \Theta(P(\delta))$. Fix any $\delta_1 \in (\underline{\delta}, \underline{\delta} + \varepsilon)$. Now suppose there is a $\delta > \underline{\delta}$ with $\hat{\Theta}(\hat{P}(\delta)) \leq \Theta(P(\delta))$. Let δ_2 denote the smallest such δ. Then

$$\begin{aligned}\log \hat{\Theta}(\hat{P}(\delta_2)) - \log \hat{\Theta}(\hat{P}(\delta_1)) &= \int_{\delta_1}^{\delta_2} \frac{\partial \log \hat{\Theta}(\hat{P}(\delta))}{\partial \delta} d\delta \\ &> \int_{\delta_1}^{\delta_2} \frac{\partial \log \Theta(P(\delta))}{\partial \delta} d\delta = \log \Theta(P(\delta_2)) - \log \Theta(P(\delta_1)),\end{aligned}$$

where the inequality uses the result from the previous paragraph that $\frac{\partial \log \hat{\Theta}(\hat{P}(\delta))}{\partial \delta} > \frac{\partial \log \Theta(P(\delta))}{\partial \delta}$ for all $\delta \in [\delta_1, \delta_2]$ since $\hat{\Theta}(\hat{P}(\delta)) \geq \Theta(P(\delta))$ by construction. But since $\hat{\Theta}(\hat{P}(\delta_1)) > \Theta(P(\delta_1))$, this implies $\hat{\Theta}(\hat{P}(\delta_2)) > \Theta(P(\delta_2))$, a contradiction. This proves $\hat{\Theta}(\hat{P}(\delta)) > \Theta(P(\delta))$ for all $\delta > \underline{\delta}$ and so, using the result from the previous paragraph, $\frac{\partial \log \hat{\Theta}(\hat{P}(\delta))}{\partial \delta} > \frac{\partial \log \Theta(P(\delta))}{\partial \delta}$, which is equivalent to $\hat{\Theta}(\hat{P}(\delta))/\Theta(P(\delta))$ increasing. ■

Proof of Proposition 11. The statement about liquidity, volume, and average asking price follow trivially from the results in Proposition 10. The same proposition shows also that the likelihood ratio $\hat{\Theta}(\hat{P}(\delta))/\Theta(P(\delta))$ is increasing and so the generalized density $\hat{\Theta}(\hat{P}(\delta))g(\delta)$ first order stochastically dominates $\Theta(P(\delta))g(\delta)$. This implies that the average transaction price also increases with the subsidy program. ■

Online Appendix

Individual's Problem: Details

For any period t, history s^{t-1}, and quality $j \in \{1, \ldots, J\}$, let $k_{i,j,t}(s^{t-1})$ denote individual i's beginning-of-period t holdings of quality j trees. For any period t, history s^t, quality $j \in \{1, \ldots, J\}$, and set $P \subset \mathbb{R}_+$, let $q_{i,j,t}(P; s^t)$ denote his net purchase in period t of quality j trees at a price $p \in P$. The individual chooses a history-contingent sequence for consumption $c_{i,t}(s^t)$ and measures of tree holdings $k_{i,j,t+1}(s^t)$ and net tree purchases $q_{i,j,t}(P; s^t)$ to maximize his expected lifetime utility

$$\sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{\tau=0}^{t-1} \pi_{s_\tau} \beta_{s_\tau} \right) \pi_{s_t} c_{i,t}(s^t).$$

This states that the individual maximizes the expected discounted value of consumption, given the stochastic process for the discount factor. The individual faces a standard budget constraint,

$$\sum_{j=1}^{J} \delta_j k_{i,j,t}(s^{t-1}) = c_{i,t}(s^t) + \int_0^\infty p \left(\sum_{j=1}^{J} q_{i,j,t}(\{p\}; s^t) \right) dp,$$

for all t and s^t. The left hand side is the fruit produced by the trees he owns at the start of period t. The right hand side is consumption plus the net purchase of trees at nonnegative prices p. He also faces a law of motion for his tree holdings,

$$k_{i,j,t+1}(s^t) = k_{i,j,t}(s^{t-1}) + q_{i,j,t}(\mathbb{R}_+; s^t),$$

for all $j \in \{1, \ldots, J\}$. This states that the increase in his tree holdings is given by his net purchase of that quality tree. Finally, the individual faces a set of constraints that depends on whether his discount factor is high or low.

If the individual has a high discount factor, $s_t = h$, he is a buyer, which implies $q_{i,j,t}(P; s^t)$ is nonnegative for all $j \in \{1, \ldots, J\}$ and $P \subset \mathbb{R}_+$. In addition, he must have enough fruit to purchase trees,

$$\sum_{j=1}^{J} \delta_j k_{i,j,t}(s^{t-1}) \geq \int_0^\infty \max\{\Theta(p), 1\} p \left(\sum_{j=1}^{J} q_{i,j,t}(\{p\}; s^t) \right) dp.$$

If the individual wishes to purchase q trees at a price p and $\Theta(p) > 1$, he will be rationed and so must bring $\Theta(p)pq$ fruit to the market to make this purchase. This constrains his ability to buy trees in markets with excess demand. Together with the budget constraint, this also ensures consumption is nonnegative. Finally, he can only purchase quality j trees at a price p if individuals are selling them at that price, that is

$$q_{i,j,t}(P;s^t) = \int_P \gamma_j(p) \left(\sum_{j'=1}^{J} q_{i,j',t}(\{p\};s^t) \right) dp$$

for all $j \in \{1,\ldots,J\}$ and $P \subset \mathbb{R}_+$. The left hand side is the quantity of quality j trees purchased at a price $p \in P$. The integrand on the right hand side is the product of quantity of trees purchased at price p and the share of those trees that are of quality j.

If the individual has a low discount factor, $s_t = l$, he is a seller, which implies $q_{i,j,t}(P;s^t)$ is nonpositive for all $j \in \{1,\ldots,J\}$ and $P \subset \mathbb{R}_+$. In addition, he may not try to sell more trees than he owns:

$$k_{i,j,t}(s^{t-1}) \geq -\int_0^\infty \max\{\Theta(p)^{-1},1\} q_{i,j,t}(\{p\};s^t)dp,$$

for all $j \in \{1,\ldots,J\}$. Each tree only sells with probability $\min\{\Theta(p),1\}$ at price p, so if $\Theta(p) < 1$, an individual must bring $\Theta(p)^{-1}$ trees to the market to sell one of them. Sellers are not restricted from selling trees in the wrong market. Instead, in equilibrium they will be induced not to do so.

Let $\bar{V}^*(\{k_j\})$ be the supremum of the individuals' expected lifetime utility over feasible policies, given initial tree holding vector $\{k_j\}$. We prove in Proposition 1 that the function $\bar{V}^*$ satisfies the following functional equation:

$$\bar{V}(\{k_j\}) = \pi_h V_h(\{k_j\}) + \pi_l V_l(\{k_j\}), \tag{27}$$

where

$$V_h(\{k_j\}) = \max_{\{q_j,k_j'\}} \left(\sum_{j=1}^{J} \delta_j k_j - \int_0^\infty p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp + \beta_h \bar{V}(\{k_j'\}) \right) \tag{28}$$

$$\begin{aligned}
\text{subject to } & k_j' = k_j + q_j(\mathbb{R}_+) \text{ for all } j \in \{1,\ldots,J\} \\
& \sum_{j=1}^{J} \delta_j k_j \geq \int_0^\infty \max\{\Theta(p),1\} p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp, \\
& q_j(P) = \int_P \gamma_j(p) \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp \text{ for all } j \in \{1,\ldots,J\} \text{ and } P \subset \mathbb{R}_+ \\
& q_j(P) \geq 0 \text{ for all } j \in \{1,\ldots,J\} \text{ and } P \subset \mathbb{R}_+,
\end{aligned}$$

and

$$V_l(\{k_j\}) = \max_{\{q_j, k_j'\}} \left(\sum_{j=1}^{J} \delta_j k_j - \int_0^\infty p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp + \beta_l \bar{V}(\{k_j'\}) \right) \tag{29}$$

$$\begin{aligned} \text{subject to } & k_j' = k_j + q_j(\mathbb{R}_+) \text{ for all } j \in \{1, \ldots, J\} \\ & k_j \geq - \int_0^\infty \max\{\Theta(p)^{-1}, 1\} q_j(\{p\}) dp \text{ for all } j \in \{1, \ldots, J\}, \\ & q_j(P) \leq 0 \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+, \end{aligned}$$

We now prove Proposition 1 working with the recursive version of the individuals' problem.

Let $\bar{\Theta}(p) \equiv \max\{\Theta(p), 1\}$ and $\underline{\Theta}(p) = \min\{\Theta(p), 1\}$. Fix Θ and Γ and take any positive-valued numbers $\{v_{s,j}\}$ and λ that solve the Bellman equations (1), (3), and (4) for $s = l, h$. Let p_h be an optimal price for buying trees,

$$p_h \in \arg\max_p \left(\bar{\Theta}(p)^{-1} \left(\frac{\beta_h \sum_{j=1}^J \gamma_j(p) \bar{v}_j}{p} - 1 \right) \right).$$

Similarly let $p_{l,j}$ be an optimal price for selling quality j trees,

$$p_{l,j} = \arg\max_p \underline{\Theta}(p) \big(p - \beta_l \bar{v}_j\big)$$

for all δ. We seek to prove that $\bar{V}^*(\{k_j\}) \equiv \sum_{j=1}^J \bar{v}_j k_j$ where $\bar{v}_j = \pi_h v_{h,j} + \pi_l v_{l,j}$.

If $\lambda = 1$, equations (1) and (3) imply

$$\bar{v}_j = \pi_h \big(\delta_j + \beta_h \bar{v}_j\big) + \pi_l \big(\delta_j + \underline{\Theta}(p_{l,j}) p_{l,j} + (1 - \underline{\Theta}(p_{l,j})) \beta_l \bar{v}_j\big).$$

for all δ. Equivalently,

$$\bar{v}_j = \frac{\delta_j + \pi_l \underline{\Theta}(p_{l,j}) p_{l,j}}{1 - \pi_h \beta_h - \pi_l \beta_l (1 - \underline{\Theta}(p_{l,j}))} > 0.$$

Alternatively, if $\lambda > 1$, the same equations imply

$$\begin{aligned} \bar{v}_j = \pi_h & \left(\delta_j \left(\big(1 - \bar{\Theta}(p_h)^{-1}\big) + \bar{\Theta}(p_h)^{-1} \frac{\beta_h \sum_{j'=1}^J \gamma_{j'}(p_h) \bar{v}_{j'}}{p_h} \right) + \beta_h \bar{v}_j \right) \\ & + \pi_l \big(\delta_j + \underline{\Theta}(p_{l,j}) p_{l,j} + (1 - \underline{\Theta}(p_{l,j})) \beta_l \bar{v}_j\big) \end{aligned}$$

for all δ. Since $v_{l,j}$ and $v_{h,j}$ are positive by assumption so is $\bar{v}_j$, and equivalently we can write

$$\bar{v}_j \left(1 - \pi_h\beta_h - \pi_l\beta_l(1 - \underline{\Theta}(p_{l,j})) - \pi_h\beta_h\bar{\Theta}(p_h)^{-1}\frac{\delta_j \sum_{j'=1}^{J} \gamma_{j'}(p_h)\bar{v}_{j'}}{p_h\bar{v}_j}\right)$$
$$= \pi_h\delta_j\left(1 - \bar{\Theta}(p_h)^{-1}\right) + \pi_l\big(\delta_j + \underline{\Theta}(p_{l,j})p_{l,j}\big).$$

The right hand side of this expression is positive for all j. Once again since $\bar{v}_j > 0$, with $\lambda > 1$, this holds if and only if

$$1 - \pi_h\beta_h - \pi_l\beta_l(1 - \underline{\Theta}(p_{l,j})) > \pi_h\beta_h\bar{\Theta}(p_h)^{-1}\frac{\delta_j \sum_{j'=1}^{J} \gamma_{j'}(p_h)\bar{v}_{j'}}{p_h\bar{v}_j}. \quad (30)$$

If this restriction fails at any prices p_h and $p_{l,j}$, it is possible for an individual to obtain unbounded expected utility by buying and selling trees at the appropriate prices. We are interested in cases in which it is satisfied.

Next, let $\bar{V}(\{k_j\}) = \sum_{j=1}^{J} \bar{v}_j k_j$ and $V_s(\{k_j\}) \equiv \sum_{j=1}^{J} v_{s,j}k_j$ for $s = l, h$. It is easy to prove that $\bar{V}$ and $\bar{V}_s$ solve equations (27), (28), and (29) and that the same policy is optimal.

Finally, we adapt Theorem 4.3 from Werning (2009), which states the following: suppose $\bar{V}(k)$ for all k satisfies the recursive equations (27), (28), and (29) and there exists a plan that is optimal given this value function which gives rise to a sequence of tree holdings $\{k^*_{i,j,t}(s^{t-1})\}$ satisfying

$$\lim_{t\to\infty} \sum_{s^t} \left(\prod_{\tau=0}^{t-1} \pi_{s_\tau}\beta_{s_\tau}\right) \bar{V}(\{k^*_{i,j,t}(s^{t-1})\}) = 0. \quad (31)$$

Then, $\bar{V}^* = \bar{V}$.

If $\lambda = 1$, an optimal plan is to sell quality j trees at price $p_{l,j}$ when impatient and not to purchase trees when patient. This gives rise to a non-increasing sequence for tree holdings. Given the linearity of $\bar{V}$, condition (31) holds trivially.

If $\lambda > 1$, it is still optimal to sell quality j trees at price $p_{l,j}$ when impatient, but patient individuals purchase trees at price p_h and do not consume. Thus

$$k'_{h,j} = k_j + \bar{\Theta}(p_h)^{-1}\gamma_j(p_h)\frac{\sum_{j'=1}^{J} \delta_{j'}k_{j'}}{p_h}$$
$$k'_{l,j} = \big(1 - \underline{\Theta}(p_{l,j})\big)k_j.$$

Using linearity of the value function, the expected discounted value next period of an individual with tree holdings $\{k_j\}$ this period is

$$\sum_{j=1}^{J} \bar{v}_j \big(\pi_h \beta_h k'_{h,j} + \pi_l \beta_l k'_{l,j}\big)$$
$$= \sum_{j=1}^{J} \bar{v}_j \left(\pi_h \beta_h \left(k_j + \bar{\Theta}(p_h)^{-1} \gamma_j(p_h) \frac{\sum_{j'=1}^{J} \delta_{j'} k_{j'}}{p_h}\right) + \pi_l \beta_l \big(1 - \underline{\Theta}(p_{l,j})\big) k_j\right)$$
$$= \sum_{j=1}^{J} \bar{v}_j k_j \left(\pi_h \beta_h + \pi_l \beta_l \big(1 - \underline{\Theta}(p_{l,j})\big) + \pi_h \beta_h \bar{\Theta}(p_h)^{-1} \frac{\delta_j \sum_{j'=1}^{J} \gamma_{j'}(p_h) \bar{v}_{j'}}{p_h \bar{v}_j}\right),$$

where the second equality simply rearranges terms in the summation. Equation (30) implies that each term of this sum is strictly smaller than $\bar{v}_j k_j$. This implies that there exists an $\eta < 1$ such that

$$\eta > \frac{\sum_{j=1}^{J} \bar{v}_j \big(\pi_h \beta_h k'_{h,j} + \pi_l \beta_l k_{l,j}\big)}{\sum_{j=1}^{J} \bar{v}_j k_j} = \frac{\pi_h \beta_h \bar{V}(\{k'_{h,j}\}) + \pi_l \beta_l \bar{V}(\{k'_{l,j}\})}{\bar{V}(\{k_j\})},$$

and so condition (31) holds.